Journal of Nuclear Energy, Vol. 26, pp. 251 to 269. Pergamon Press 1972. Printed in Northern Ireland

TWO-GROUP NEUTRON-TRANSPORT THEORY: HALF-RANGE ORTHOGONALITY, NORMALIZATION INTEGRALS, APPLICATIONS AND COMPUTATIONS

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(*Received* 28 *September* 1971)

Abstract—A half-range orthogonality theorem concerning the established elementary solutions of the two-group neutron transport equation for isotropic scattering and plane geometry is proved. The orthogonality relations are based on a Chandrasekhar-type **H**-matrix, and all necessary normalization integrals are evaluated so that the desired expansion coefficients may be expressed concisely in terms of inner products. The half-range orthogonality theorem is used to construct tractable solutions to typical half-space problems. In addition, the required **H**-matrix is calculated to benchmark accuracy, and explicit results for several quantities of interest are reported for the Milne, albedo and constant-source problems.

1. INTRODUCTION

ONE OF the principal merits of the singular-eigenfunction-expansion technique, introduced initially by CASE (1960) in regard to one-speed neutron-transport theory, is the systematic and classical manner in which solutions to boundary-value problems in particle transport theory are constructed. In general, we first construct a sufficiently general set of solutions, denoted as normal modes, to the homogeneous equation of transfer. To a resulting superposition (with arbitrary expansion coefficients) we add a particular solution to account for any inhomogeneous source term, and we then constrain the complete solution to meet the boundary conditions of a specified problem. At this point, a completeness theorem (either full- or half-range) is proved, which in fact ensures that a sufficiently general set of normal modes has been obtained. In many cases (CASE and ZWEIFEL, 1967) the various completeness proofs are actually constructive, in that, in addition to proving an expansion theorem, we can construct analytical results for the desired expansion coefficients.

KUŠČER *et al.* (1964) were the first to observe that the results of the half-range completeness theorem related to one-speed theory could be expressed quite naturally in terms of scalar products based on the proof of a half-range orthogonality theorem. Though the results so expressed were, of course, identical to those patiently deduced from the completeness theorem, the use of orthogonality relations provided a significant impetus to a more universal appeal of the singular-eigenfunction method.

We should like to demonstrate here the manner in which an orthogonality theorem for the two-group model can be used to codify the results of half-range normal-mode expansions. Half-range orthogonality relations related to the generalized picket-fence model in radiative transfer, a special case of multi-group neutron transport theory, have been reported by SIEWERT and ZWEIFEL (1966). There, because of the rather special structure of the picket-fence model, SIEWERT and ZWEIFEL (1966) obtained exact closed-form results and were able to express the half-range weight matrix solely in terms of scalar X- or H-functions (SIEWERT and ÖZIŞIK, 1969). For the general two-group model considered here, the weight function is written in terms of an **H**-matrix for which no quadrature expression is available though existence and uniqueness theorems (SIEWERT *et al.*, 1972) and rapidly convergent computational methods are available. We note that similar half-range results have been reported (SIEWERT, 1971) for CHANDRASEKHAR's (1950) model for the scattering of polarized light. Further, PAHOR and SHULTIS (1969), BOWDEN and MCCROSSON (1971) and METCALF and ZWEIFEL (1968) have all contributed to the literature on half-space applications concerning multi-group neutron-transport theory. For the two-group model, we believe the current approach is more satisfactory because: (i) In the classical mathematical manner, it has been argued (SIEWERT *et al.*, 1972) that the relevant normal modes form a complete and orthogonal half-range basis for the expansion of Hölder two-vectors. (ii) The appropriate existence and uniqueness theorems required to complete the analysis of BOWDEN and MCCROSSON (1971) have been established (SIEWERT *et al.*, 1972) for the **H**-matrix required here. (iii) Because our final results are expressed solely in terms of the **H**-matrix, which can be computed straightforwardly from a very concise regular-integral equation, we need not solve numerically any singular-integral equations, as have METCALF and ZWEIFEL (1968).

We consider the two-group transport equation written as

$$\mu \frac{\partial}{\partial x} \mathbf{I}(x, \mu) + \mathbf{\Sigma I}(x, \mu) = \mathbf{Q} \int_{-1}^{1} \mathbf{I}(x, \mu') \, d\mu', \tag{1}$$

where the elements of the two-vector $\mathbf{I}(x, \mu)$ are the group angular fluxes, μ is the direction cosine of the propagating radiation (as measured from the positive x-axis) and x is the optical variable defined (without loss of generality) in terms of the smaller of the two total group cross-sections. With the optical variable so defined, the $\mathbf{\Sigma}$-matrix is given by

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma > 1, \tag{2}$$

where σ denotes the ratio of the two total cross-sections. In addition, the transfer matrix, with arbitrary real non-negative elements $q_{\alpha\beta}$, for isotropic scattering is denoted by $\mathbf{Q}$. If we let $\mathbf{P}$ be a 2×2 matrix with elements $p_{\alpha\beta} = (q_{2\alpha}/q_{\alpha 2})^{1/2}\delta_{\alpha\beta}$, then clearly $\mathbf{C} = \mathbf{PQP}^{-1}$ is symmetric, and thus equation (1) for $\mathbf{I}(x, \mu) = \mathbf{P}^{-1}\mathbf{\Psi}(x, \mu)$ can be pre-multiplied by $\mathbf{P}$ to yield

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Sigma \Psi}(x, \mu) = \mathbf{C} \int_{-1}^{1} \mathbf{\Psi}(x, \mu') \, d\mu', \tag{3}$$

the form we prefer since the new transfer matrix $\mathbf{C}$ is symmetric. Here we consider that the $\mathbf{Q}$-matrix is neither upper nor lower triangular since these two triangular cases can be solved in terms of the one-speed theory. We also excluded the case $\det \mathbf{Q} = 0$ which has been solved by SIEWERT and ZWEIFEL (1966).

Since the elementary solutions of equation (3) have been reported (SIEWERT and ZWEIFEL, 1966), we simply summarize the results here in order to establish the required formalism and notation. Seeking solutions of equation (3) of the form

$$\mathbf{\Psi}_\xi(x, \mu) = \mathbf{F}(\xi, \mu) e^{-x/\xi} \tag{4}$$

yields

$$(\xi \mathbf{\Sigma} - \mu \mathbf{I}) \mathbf{F}(\xi, \mu) = \xi \mathbf{C M}(\xi), \tag{5}$$

where **I** is the unit matrix and

$$\mathbf{M}(\xi) = \int_{-1}^{1} \mathbf{F}(\xi, \mu)\, d\mu. \tag{6}$$

Regarding the discrete spectrum, $\xi = \pm\nu_i \notin (-1, 1)$, we note that

$$\mathbf{F}(\pm\nu_i, \mu) = \nu_i \mathbf{D}(\nu_i, \pm\mu)\mathbf{C}\mathbf{M}(\nu_i), \tag{7}$$

where $\pm\nu_i$ are the zeros of the dispersion function $\Lambda(z) = \det \mathbf{\Lambda}(z)$, with

$$\mathbf{\Lambda}(z) = \mathbf{I} + z\int_{-1}^{1} \boldsymbol{\psi}(\mu) \frac{d\mu}{\mu - z}, \tag{8}$$

where the characteristic matrix is

$$\boldsymbol{\psi}(\mu) = \begin{bmatrix} \theta(\mu) & 0 \\ 0 & 1 \end{bmatrix} \mathbf{C} = \mathbf{\Theta}(\mu)\mathbf{C}, \tag{9}$$

with $\theta(\mu) = 1$, $\mu \in (-1/\sigma, 1/\sigma)$ and $\theta(\mu) = 0$, otherwise. In addition

$$\mathbf{D}(\xi, \mu) = \begin{bmatrix} \dfrac{1}{\sigma\xi - \mu} & 0 \\ 0 & \dfrac{1}{\xi - \mu} \end{bmatrix} \tag{10}$$

and

$$\mathbf{\Lambda}(\nu_i)\mathbf{M}(\nu_i) = \mathbf{0}. \tag{11}$$

Siewert and Shieh (1967) have summarized the number and types of zeros of the dispersion function; we note that there is either one pair $\pm\nu_1$, or two pairs $\pm\nu_1$ and $\pm\nu_2$; here ν_i denotes a discrete eigenvalue with a positive real (imaginary) part. We use κ to denote the number of pairs of discrete eigenvalues. Equation (8) may be expanded to yield a more explicit form of the dispersion function:

$$\Lambda(z) = 1 - 2c_{11}z\tau\left(\frac{1}{\sigma z}\right) - 2c_{22}z\tau\left(\frac{1}{z}\right) + 4Cz^2\tau\left(\frac{1}{z}\right)\tau\left(\frac{1}{\sigma z}\right), \tag{12}$$

where we have introduced the abbreviations $C = \det \mathbf{C}$ and $\tau(x) = \tanh^{-1} x$. In the limit $|z| \to \infty$, equation (12) yields

$$\Lambda(\infty) = 1 - \frac{2}{\sigma} c_{11} - 2c_{22} + \frac{4}{\sigma} C. \tag{13}$$

For the continuum $\xi = \nu \in (-1, 1)$, we write the solution to equation (5) as

$$\mathbf{F}(\nu, \mu) = [\nu\mathbf{K}(\nu, \mu) + \omega(\nu)\, \boldsymbol{\delta}(\nu, \mu)]\mathbf{C}\mathbf{M}(\nu), \tag{14}$$

where

$$\mathbf{K}(\nu, \mu) = \begin{bmatrix} Pv\left(\dfrac{1}{\sigma\nu - \mu}\right) & 0 \\ 0 & Pv\left(\dfrac{1}{\nu - \mu}\right) \end{bmatrix} \quad \text{and} \quad \boldsymbol{\delta}(\nu, \mu) = \begin{bmatrix} \delta(\sigma\nu - \mu) & 0 \\ 0 & \delta(\nu - \mu) \end{bmatrix}, \tag{15}$$

where the distribution $Pv(1/x)$ denotes that ensuing integrals are to be evaluated in the Cauchy principal-value sense, and $\delta(x)$ is the Dirac delta distribution. Equation (14) when integrated over μ from -1 to 1 yields

$$[\boldsymbol{\lambda}(\nu) - \omega(\nu)\boldsymbol{\psi}(\nu)]\mathbf{M}(\nu) = \mathbf{0}, \tag{16}$$

and therefore the parameter $\omega(\nu)$ is to be determined from the requirement

$$\det[\boldsymbol{\lambda}(\nu) - \omega(\nu)\boldsymbol{\psi}(\nu)] = 0, \tag{17}$$

where

$$\boldsymbol{\lambda}(\nu) = \mathbf{I} + \nu P\int_{-1}^{1} \boldsymbol{\psi}(\mu)\frac{d\mu}{\mu - \nu}. \tag{18}$$

If we now let

$$\text{Region ①} \Rightarrow \nu \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right)$$

$$\text{Region ②} \Rightarrow \nu \in \left(-1, -\frac{1}{\sigma}\right) U \left(\frac{1}{\sigma}, 1\right),$$

then equation (17) yields two solutions $\omega_1^{(1)}(\nu)$ and $\omega_2^{(1)}(\nu)$ for $\nu \in$ ① and one solution $\omega^{(2)}(\nu)$ for $\nu \in$ ②. We thus find

$$\mathbf{F}_\alpha^{(1)}(\nu, \mu) = [\nu\mathbf{K}(\nu, \mu) + \omega_\alpha^{(1)}(\nu)\, \boldsymbol{\delta}(\nu, \mu)]\mathbf{C}\mathbf{M}_\alpha^{(1)}(\nu), \qquad \nu \in ①, \qquad \alpha = 1, 2, \tag{19}$$

and

$$\mathbf{F}^{(2)}(\nu, \mu) = [\nu\mathbf{K}(\nu, \mu) + \omega^{(2)}(\nu)\, \boldsymbol{\delta}(\nu, \mu)]\mathbf{C}\mathbf{M}^{(2)}(\nu), \qquad \nu \in ②. \tag{20}$$

In Section 3 where all necessary normalization integrals are evaluated, we summarize the explicit forms of the normal modes reported by SIEWERT and ZWEIFEL (1966). Here we note only that our general solution to equation (3) can be written as

$$\boldsymbol{\Psi}(x,\mu) = \sum_{i=1}^{\kappa} [A(\nu_i)\mathbf{F}(\nu_i,\mu)e^{-x/\nu_i} + A(-\nu_i)\mathbf{F}(-\nu_i,\mu)e^{x/\nu_i}]$$
$$+ \int_{①} [A_1^{(1)}(\nu)\mathbf{F}_1^{(1)}(\nu,\mu) + A_2^{(1)}(\nu)\mathbf{F}_2^{(1)}(\nu,\mu)]e^{-x/\nu}\, d\nu + \int_{②} A^{(2)}(\nu)\mathbf{F}^{(2)}(\nu,\mu)e^{-x/\nu}\, d\nu, \tag{21}$$

where $A(\pm\nu_i)$, $A_1^{(1)}(\nu)$, $A_2^{(1)}(\nu)$ and $A^{(2)}(\nu)$ are the expansion coefficients to be determined once a specific problem is considered and the resulting boundary conditions specified.

2. HALF-RANGE ORTHOGONALITY

Relying on a paper by SIEWERT and ZWEIFEL (1966) for the full-range completeness and orthogonality theorems, we consider here a typical half-space problem where we seek a solution to a boundary-value constraint of the form

$$\boldsymbol{\Phi}(\mu) = \sum_{i=1}^{\kappa} A(\nu_i)\mathbf{F}(\nu_i, \mu) + \int_0^{1/\sigma} [A_1^{(1)}(\nu)\mathbf{F}_1^{(1)}(\nu, \mu)$$
$$+ A_2^{(1)}(\nu)\mathbf{F}_2^{(1)}(\nu, \mu)]\, d\nu + \int_{1/\sigma}^{1} A^{(2)}(\nu)\mathbf{F}^{(2)}(\nu, \mu)\, d\nu, \qquad \mu \in (0, 1). \tag{22}$$

Here $\mathbf{\Phi}(\mu)$ is the expansion function and is considered to be given; it may contain a diverging (at infinity) solution of the transport equation, as for the Milne problem, or a particular solution relevant to an inhomogeneous source term and/or a specified incident distribution.

SIEWERT *et al.* (1972) have argued that equation (22) is a valid expansion for arbitrary Hölder vectors $\mathbf{\Phi}(\mu)$ for all of the considered values of σ and $\mathbf{C}$ except when the two conditions $C < 0$ *and* $\Lambda(\infty) > 0$ are simultaneously imposed. It is likely that this special case may be included, but to date a definitive proof is not available. We now wish to prove an orthogonality theorem that will allow the solution to equation (22) to be written in a concise manner:

$$A(\nu_i) = \frac{1}{N(\nu_i)}[\mathbf{F}(\nu_i, \mu), \mathbf{\Phi}(\mu)], \tag{23a}$$

$$A_\alpha^{(1)}(\nu) = \frac{1}{N^{(1)}(\nu)}[\mathbf{F}_\alpha^{(1)}(\nu, \mu), \mathbf{\Phi}(\mu)], \qquad \nu \in \left(0, \frac{1}{\sigma}\right), \qquad \alpha = 1 \text{ and } 2, \tag{23b}$$

$$A^{(2)}(\nu) = \frac{1}{N^{(2)}(\nu)}[\mathbf{F}^{(2)}(\nu, \mu), \mathbf{\Phi}(\mu)], \qquad \nu \in \left(\frac{1}{\sigma}, 1\right), \tag{23c}$$

where $[X, Y]$ is an appropriate inner product and $N(\nu_i)$ and $N^{(\alpha)}(\nu)$ are the normalization factors.

Theorem: *The eigenvectors* $\mathbf{F}(\nu_i, \mu)$, $\mathbf{F}_1^{(1)}(\nu, \mu)$, $\mathbf{F}_2^{(1)}(\nu, \mu)$ *and* $\mathbf{F}^{(2)}(\nu, \mu)$, $\nu \in (0, 1)$, *are orthogonal on the half-range*, $\mu \in (0, 1)$, *to the related set* $\mathbf{G}(\nu_i, \mu)$, $\mathbf{G}_1^{(1)}(\nu, \mu)$, $\mathbf{G}_2^{(1)}(\nu, \mu)$ *and* $\mathbf{G}^{(2)}(\nu, \mu)$, $\nu \in (0, 1)$, *in the sense that*

$$\int_0^1 \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu \, d\mu = 0, \qquad \xi \neq \xi'; \qquad \xi, \xi' = \nu_i \text{ or } \in (0, 1). \tag{24}$$

Here

$$\mathbf{G}(\xi, \mu) = \mathbf{E}(\xi, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\xi)\left(\mathbf{\Sigma} - \frac{\mu}{\xi}\mathbf{I}\right)\mathbf{F}(\xi, \mu), \qquad \xi = \nu_i \text{ or } \in (0, 1), \tag{25}$$

where, in general, we write

$$\mathbf{E}(\xi, \mu) = \begin{bmatrix} \xi Pv\left(\dfrac{1}{\sigma\xi - \mu}\right) & 0 \\ 0 & \xi Pv\left(\dfrac{1}{\xi - \mu}\right) \end{bmatrix} + \omega(\xi)\,\boldsymbol{\delta}(\xi, \mu). \tag{26}$$

In addition $\mathbf{H}(z)$ is the $\mathbf{H}$-matrix, and $\mathbf{h}(z)$ is defined by

$$\mathbf{h}(z) = \begin{bmatrix} H_{11}(z/\sigma) & H_{12}(z/\sigma) \\ H_{21}(z) & H_{22}(z) \end{bmatrix}, \tag{27}$$

where $H_{\alpha\beta}(z)$ is used to denote the elements of $\mathbf{H}(z)$. The superscript tilde is used to denote the transpose operation.

Though CHANDRASEKHAR's (1950) invariance principles may be used in the manner of PAHOR (1966) to develop the equations

$$\tilde{\mathbf{H}}(\mu)\mathbf{\lambda}(\mu) = \mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\nu)\mathbf{\psi}(\nu)\frac{d\nu}{\nu - \mu}, \qquad \mu \in (0, 1/\sigma), \tag{28a}$$

$$\tilde{\mathbf{H}}(\mu)\mathbf{\lambda}(\mu)\mathbf{M}^{(2)}(\mu) = \left[\mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\nu)\mathbf{\psi}(\nu)\frac{d\nu}{\nu - \mu}\right]\mathbf{M}^{(2)}(\mu), \qquad \mu \in (1/\sigma, 1), \tag{28b}$$

and

$$\left[\mathbf{I} + \nu_i \int_0^1 \tilde{\mathbf{H}}(\nu)\Psi(\nu)\frac{d\nu}{\nu - \nu_i}\right]\mathbf{M}(\nu_i) = \mathbf{0}, \qquad i = 1, 2, \ldots, \kappa, \tag{28c}$$

sufficient to specify the H-matrix for use with the invariance principles, such a procedure is by necessity limited to non-multiplying media. However, since SIEWERT *et al.* (1972) have proved that equations (28) yield a unique solution for all considered values of σ and the transfer matrix **C**, except the non-multiplying case with $C < 0$, and since equation (22) is mathematically a valid expansion even for multiplying media, we choose to use equations (28) to specify our **H**-matrix for all considered cases, both non-multiplying and multiplying. This view is considered quite natural since, as we shall show, equations (28) yield an **H**-matrix which can be used in our orthogonality theorem to yield a valid solution to equation (22) for all cases considered. We also use the definition of SIEWERT *et al.* (1972) for extending the **H**-matrix to the complex plane and thus write

$$\tilde{\mathbf{H}}(z)\mathbf{\Lambda}(z) = \mathbf{I} + z\int_0^1 \tilde{\mathbf{H}}(\mu)\Psi(\mu)\frac{d\mu}{\mu - z} \tag{29a}$$

or

$$\mathbf{H}(z) = \mathbf{I} + z\mathbf{H}(z)\mathbf{C}\int_0^1 \tilde{\mathbf{H}}(\mu)\,\mathbf{\Theta}(\mu)\frac{d\mu}{\mu + z} \tag{29b}$$

and the subsequent identity

$$\mathbf{H}(z)\mathbf{C}\tilde{\mathbf{H}}(-z)\mathbf{\Lambda}(z) = \mathbf{C}, \tag{30}$$

which allows equation (28c) to be written as

$$\mathbf{H}^{-1}(-\nu_i)\mathbf{C}\mathbf{M}(\nu_i) = \mathbf{0}, \;\; i = 1, 2, \ldots, \kappa. \tag{31}$$

We now proceed to establish a proof of our half-range orthogonality theorem. Equation (5) can be pre-multiplied by $1/\xi\tilde{\mathbf{G}}(\xi', \mu)$, where from equation (25) we conclude that

$$\tilde{\mathbf{G}}(\xi', \mu) = \tilde{\mathbf{M}}(\xi')\mathbf{C}\tilde{\mathbf{H}}^{-1}(\xi')\tilde{\mathbf{h}}(\mu)\mathbf{E}(\xi', \mu), \;\; \xi' = \nu_i \;\text{ or }\; \in (0, 1), \tag{32}$$

and integrated over μ from 0–1 to yield

$$\frac{1}{\xi}\int_0^1 \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu\, d\mu$$
$$= \tilde{\mathbf{M}}(\xi')\mathbf{C}\tilde{\mathbf{H}}^{-1}(\xi')\left[\int_0^1 \tilde{\mathbf{h}}(\mu)\mathbf{E}(\xi', \mu)\mathbf{E}(\xi, \mu)\, d\mu\mathbf{\Sigma} - \int_0^1 \tilde{\mathbf{h}}(\mu)\mathbf{E}(\xi', \mu)\, d\mu\right]\mathbf{C}\mathbf{M}(\xi). \tag{33}$$

We can now post-multiply equation (32) by $\mu/\xi'\ \mathbf{F}(\xi, \mu)$, integrate over μ from 0–1 and subtract the resulting equation from equation (33) to obtain

$$\left(\frac{1}{\xi} - \frac{1}{\xi'}\right)\int_0^1 \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu\, d\mu = \tilde{\mathbf{M}}(\xi')\mathbf{C}\tilde{\mathbf{H}}^{-1}(\xi')[\mathbf{Y}(\xi) - \mathbf{Y}(\xi')]\mathbf{C}\mathbf{M}(\xi), \tag{34}$$

after some partial-fraction analysis. In equation (34) we have introduced the matrix

$$\mathbf{Y}(\xi) = \int_0^1 \tilde{\mathbf{h}}(\mu)\mathbf{E}(\xi, \mu)\, d\mu, \tag{35}$$

which after a change of the integration variable can be expressed as

$$\mathbf{Y}(\xi) = \int_0^1 \tilde{\mathbf{H}}(\mu)\,\mathbf{\Theta}(\mu)\left[\xi Pv\left(\frac{1}{\xi-\mu}\right) + \omega(\xi)\,\delta(\xi-\mu)\right] \mathrm{d}\mu. \tag{36}$$

Equations (28) which define the **H**-matrix along with the subsequent equations (29), (30) and (31) can now be used with equations (11), (16) and (36) to show for all ξ and $\xi' = \nu_i$ or $\in (0, 1)$ that the right-hand side of equation (34) vanishes identically. Thus

$$\left(\frac{1}{\xi} - \frac{1}{\xi'}\right)\int_0^1 \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu\,\mathrm{d}\mu = 0; \qquad \xi, \xi' = \nu_i \quad \text{or} \quad \in (0, 1), \tag{37}$$

which proves the theorem. We note that a similar proof has been reported in more detail by SIEWERT (1971) for a model of the scattering of polarized light.

3. NORMALIZATION INTEGRALS

The half-range orthogonality theorem has been established, and we would therefore like to evaluate the necessary normalization integrals, so that all expansion coefficients appearing in equation (22) can be expressed concisely in terms of integrals of the expansion function $\mathbf{\Phi}(\mu)$.

Though the full-range orthogonality relations have been reported by SIEWERT and SHIEH (1967), we would like to summarize both full-range and half-range cases here in terms of the general forms of $\mathbf{F}(\xi, \mu)$ given by equations (7), (19) and (20). For the full-range, we write

$$\int_{-1}^1 \tilde{\mathbf{F}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu\,\mathrm{d}\mu = 0, \qquad \xi \neq \xi'; \qquad \xi, \xi' = \pm\nu_i \quad \text{or} \quad \in (-1, 1), \tag{38a}$$

whereas for the half-range

$$\int_0^1 \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu\,\mathrm{d}\mu = 0, \qquad \xi \neq \xi'; \qquad \xi, \xi' = \nu_i \quad \text{or} \quad \in (0, 1). \tag{38b}$$

Here $\mathbf{F}(\xi, \mu)$ is any solution of equation (5) and

$$\mathbf{G}(\xi, \mu) = \mathbf{E}(\xi, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\xi)\left(\mathbf{\Sigma} - \frac{\mu}{\xi}\mathbf{I}\right)\mathbf{F}(\xi, \mu). \tag{39}$$

Considering the full-range normalization integrals, we find

$$\int_{-1}^1 \tilde{\mathbf{F}}(\pm\nu_i, \mu)\mathbf{F}(\pm\nu_i, \mu)\mu\,\mathrm{d}\mu = \pm S(\nu_i), \qquad i = 1, 2, \ldots, \kappa, \tag{40a}$$

$$\int_{-1}^1 \tilde{\mathbf{F}}_\alpha^{(1)}(\nu', \mu)\mathbf{F}_\beta^{(1)}(\nu, \mu)\mu\,\mathrm{d}\mu = S_\alpha^{(1)}(\nu)\,\delta(\nu - \nu')\,\delta_{\alpha\beta}; \qquad \nu, \nu' \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right); \qquad \alpha, \beta = 1, 2, \tag{40b}$$

and

$$\int_{-1}^1 \tilde{\mathbf{F}}^{(2)}(\nu', \mu)\mathbf{F}^{(2)}(\nu, \mu)\mu\,\mathrm{d}\mu = S^{(2)}(\nu)\,\delta(\nu - \nu'); \qquad \nu, \nu' \in \left(-1, -\frac{1}{\sigma}\right) U\left(\frac{1}{\sigma}, 1\right), \tag{40c}$$

whereas for the half-range we conclude that

$$\int_0^1 \tilde{\mathbf{G}}(\nu_i, \mu)\mathbf{F}(\nu_i, \mu)\mu \, \mathrm{d}\mu = S(\nu_i), \qquad i = 1, 2, \ldots, \kappa, \tag{41a}$$

$$\int_0^1 \tilde{\mathbf{G}}_\alpha^{(1)}(\nu', \mu)\mathbf{F}_\beta^{(1)}(\nu, \mu)\mu \, \mathrm{d}\mu = S_\alpha^{(1)}(\nu)\, \delta(\nu - \nu')\, \delta_{\alpha\beta}; \qquad \nu, \nu' \in \left(0, \frac{1}{\sigma}\right), \qquad \alpha, \beta = 1, 2, \tag{41b}$$

and

$$\int_0^1 \tilde{\mathbf{G}}^{(2)}(\nu', \mu)\mathbf{F}^{(2)}(\nu, \mu)\mu \, \mathrm{d}\mu = S^{(2)}(\nu)\, \delta(\nu - \nu'); \qquad \nu, \nu' \in \left(\frac{1}{\sigma}, 1\right). \tag{41c}$$

Here

$$S(\nu_i) = \nu_i^2 \tilde{\mathbf{M}}(\nu_i)\mathbf{C} \frac{\mathrm{d}}{\mathrm{d}z} \mathbf{\Lambda}(z) \bigg|_{z=\nu_i} \mathbf{M}(\nu_i), \tag{42a}$$

$$S_\alpha^{(1)}(\nu) = \nu[\omega_\alpha^{(1)}(\nu)\omega_\alpha^{(1)}(\nu) + \pi^2\nu^2]\tilde{\mathbf{M}}_\alpha^{(1)}(\nu)\tilde{\boldsymbol{\psi}}(\nu)\boldsymbol{\psi}(\nu)\mathbf{M}_\alpha^{(1)}(\nu), \tag{42b}$$

and

$$S^{(2)}(\nu) = \nu[\omega^{(2)}(\nu)\omega^{(2)}(\nu) + \pi^2\nu^2]\tilde{\mathbf{M}}^{(2)}(\nu)\tilde{\boldsymbol{\psi}}(\nu)\boldsymbol{\psi}(\nu)\mathbf{M}^{(2)}(\nu). \tag{42c}$$

We note again, for emphasis, that here **C** is considered symmetric.

Though the representations of the two eigenvectors given by equation (19) for $\nu \in$ Region ① are convenient for proving completeness and orthogonality theorems, we prefer for actual applications the linear combinations

$$\mathbf{\Phi}_\alpha^{(1)}(\nu, \mu) = T_{\alpha 1}(\nu)\mathbf{F}_1^{(1)}(\nu, \mu) + T_{\alpha 2}(\nu)\mathbf{F}_2^{(1)}(\nu, \mu) \tag{43}$$

which yield the explicit forms developed by SIEWERT and ZWEIFEL (1966):

$$\mathbf{\Phi}_1^{(1)}(\nu, \mu) = \begin{bmatrix} c_{11}\nu Pv\left(\dfrac{1}{\sigma\nu - \mu}\right) + \lambda_{11}(\nu)\, \delta(\sigma\nu - \mu) \\ c_{21}\nu Pv\left(\dfrac{1}{\nu - \mu}\right) + \lambda_{21}(\nu)\, \delta(\nu - \mu) \end{bmatrix} \tag{44a}$$

and

$$\mathbf{\Phi}_2^{(1)}(\nu, \mu) = \begin{bmatrix} c_{12}\nu Pv\left(\dfrac{1}{\sigma\nu - \mu}\right) + \lambda_{12}(\nu)\, \delta(\sigma\nu - \mu) \\ c_{22}\nu Pv\left(\dfrac{1}{\nu - \mu}\right) + \lambda_{22}(\nu)\, \delta(\nu - \mu) \end{bmatrix}. \tag{44b}$$

For the additional eigenvectors, we normalize equations (7) and (20) to obtain

$$\mathbf{\Phi}^{(2)}(\nu, \mu) = \begin{bmatrix} \dfrac{c_{12}\nu}{\sigma\nu - \mu} \\ \nu f(\nu) Pv\left(\dfrac{1}{\nu - \mu}\right) + \lambda(\nu)\, \delta(\nu - \mu) \end{bmatrix}, \tag{45a}$$

where

$$f(\nu) = c_{22} - 2C\nu\tau\left(\frac{1}{\sigma\nu}\right) \tag{45b}$$

and $\lambda(\nu) = \det \mathbf{\lambda}(\nu)$:

$$\lambda(\nu) = 1 - 2c_{11}\nu\tau\left(\frac{1}{\sigma\nu}\right) - 2c_{22}\nu\tau(\nu) + 4C\nu^2\tau(\nu)\tau\left(\frac{1}{\sigma\nu}\right), \tag{45c}$$

and also

$$\mathbf{\Phi}(\pm\nu_i, \mu) = \begin{bmatrix} \dfrac{c_{12}\nu_i}{\sigma\nu_i \mp \mu} \\ \dfrac{\nu_i f(\nu_i)}{\nu_i \mp \mu} \end{bmatrix}, \qquad i = 1, 2, \ldots, \kappa. \tag{45d}$$

The solutions given by equations (44), though more concise than the $\mathbf{F}_\alpha^{(1)}(\nu, \mu)$ given by equation (19), are not orthogonal for $\nu = \nu'$; however, as discussed by SIEWERT and ZWEIFEL (1966), a Schmidt-type procedure can be used to construct suitable adjoint vectors. We should now like to summarize our final results regarding the orthogonality relations:

$$\int_{-1}^{1} \tilde{\mathbf{X}}(\xi', \mu)\mathbf{\Phi}(\xi, \mu)\mu \, \mathrm{d}\mu = 0, \qquad \xi \neq \xi'; \qquad \xi, \xi' = \pm\nu_i \quad \text{or} \quad \in(-1, 1), \tag{46a}$$

$$\int_{0}^{1} \mathbf{\Theta}(\xi', \mu)\mathbf{\Phi}(\xi, \mu)\mu \, \mathrm{d}\mu = 0, \qquad \xi \neq \xi'; \qquad \xi, \xi' = \nu_i \quad \text{or} \quad \in(0, 1), \tag{46b}$$

and

$$\int_{-1}^{1} \tilde{\mathbf{X}}(\pm\nu_i, \mu)\mathbf{\Phi}(\pm\nu_i, \mu)\mu \, \mathrm{d}\mu = \pm N(\nu_i), \qquad i = 1, 2, \ldots, \kappa, \tag{47a}$$

$$\int_{-1}^{1} \tilde{\mathbf{X}}_\alpha^{(1)}(\nu', \mu)\mathbf{\Phi}_\beta^{(1)}(\nu, \mu)\mu \, \mathrm{d}\mu = N^{(1)}(\nu) \, \delta(\nu - \nu') \, \delta_{\alpha\beta}; \qquad \nu, \nu' \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right), \tag{47b}$$

$$\int_{-1}^{1} \tilde{\mathbf{X}}^{(2)}(\nu', \mu)\mathbf{\Phi}^{(2)}(\nu, \mu)\mu \, \mathrm{d}\mu = N^{(2)}(\nu) \, \delta(\nu - \nu'); \qquad \nu, \nu' \in \left(-1, -\frac{1}{\sigma}\right) U \left(\frac{1}{\sigma}, 1\right), \tag{47c}$$

and

$$\int_{0}^{1} \tilde{\mathbf{\Theta}}(\nu_i, \mu)\mathbf{\Phi}(\nu_i, \mu)\mu \, \mathrm{d}\mu = N(\nu_i), \qquad i = 1, 2, \ldots, \kappa, \tag{48a}$$

$$\int_{0}^{1} \tilde{\mathbf{\Theta}}_\alpha^{(1)}(\nu', \mu)\mathbf{\Phi}_\beta^{(1)}(\nu, \mu)\mu \, \mathrm{d}\mu = N^{(1)}(\nu) \, \delta(\nu - \nu') \, \delta_{\alpha\beta}; \qquad \nu, \nu' \in \left(0, \frac{1}{\sigma}\right), \tag{48b}$$

$$\int_{0}^{1} \tilde{\mathbf{\Theta}}^{(2)}(\nu', \mu)\mathbf{\Phi}^{(2)}(\nu, \mu)\mu \, \mathrm{d}\mu = N^{(2)}(\nu) \, \delta(\nu - \nu'); \qquad \nu, \nu' \in \left(\frac{1}{\sigma}, 1\right). \tag{48c}$$

Here

$$\mathbf{X}(\pm\nu_i, \mu) = \mathbf{\Phi}(\pm\nu_i, \mu) \tag{49a}$$

$$\mathbf{X}_1^{(1)}(\nu, \mu) = N_{22}(\nu)\mathbf{\Phi}_1^{(1)}(\nu, \mu) - N_{12}(\nu)\mathbf{\Phi}_2^{(1)}(\nu, \mu) \tag{49b}$$

$$\mathbf{X}_2^{(1)}(\nu, \mu) = N_{11}(\nu)\mathbf{\Phi}_2^{(1)}(\nu, \mu) - N_{21}(\nu)\mathbf{\Phi}_1^{(1)}(\nu, \mu) \tag{49c}$$

$$\mathbf{X}^{(2)}(\nu, \mu) = \mathbf{\Phi}^{(2)}(\nu, \mu), \tag{49d}$$

and for $\xi = \nu_i$ or $\in (0, 1)$

$$\mathbf{\Theta}(\xi, \mu) = \mathbf{E}(\xi, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\xi)\left(\mathbf{\Sigma} - \frac{\mu}{\xi}\mathbf{I}\right)\mathbf{X}(\xi, \mu). \tag{50}$$

Also

$$N_{11}(\nu) = 1 - 4c_{11}\nu\tau(\sigma\nu) + 4\nu^2[c_{11}{}^2\tau^2(\sigma\nu) + c_{12}c_{21}\tau^2(\nu)] + \pi^2\nu^2(c_{11}{}^2 + c_{12}c_{21}), \tag{51a}$$

$$N_{ij}(\nu) = c_{ij}[4c_{11}\nu^2\tau^2(\sigma\nu) + 4c_{22}\nu^2\tau^2(\nu) - 2\nu\tau(\sigma\nu) - 2\nu\tau(\nu) + \pi^2\nu^2(c_{11} + c_{22})], \quad i \neq j, \tag{51b}$$

and

$$N_{22}(\nu) = 1 - 4c_{22}\nu\tau(\nu) + 4\nu^2[c_{22}{}^2\tau^2(\nu) + c_{12}c_{21}\tau^2(\sigma\nu)] + \pi^2\nu^2(c_{22}{}^2 + c_{12}c_{21}). \tag{51c}$$

Finally, the normalization factors are given (for both full-range and half-range cases) by

$$N(\nu_i) = 2\nu_i^2\left\{c_{12}{}^2\left[\frac{\sigma\nu_i}{(\sigma\nu_i)^2 - 1} - \tau\left(\frac{1}{\sigma\nu_i}\right)\right] + \left[c_{22} - 2C\nu_i\tau\left(\frac{1}{\sigma\nu_i}\right)\right]^2\left[\frac{\nu_i}{\nu_i{}^2 - 1} - \tau\left(\frac{1}{\nu_i}\right)\right]\right\}, \tag{52a}$$

$$N^{(1)}(\nu) = \nu\Lambda^+(\nu)\Lambda^-(\nu), \qquad \nu \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right), \tag{52b}$$

$$N^{(2)}(\nu) = \nu\Lambda^+(\nu)\Lambda^-(\nu), \qquad \nu \in \left(-1, -\frac{1}{\sigma}\right) U \left(\frac{1}{\sigma}, 1\right), \tag{52c}$$

where $\Lambda^{\pm}(\nu)$ denotes the limiting values as the branch cut of $\Lambda(z)$ is approached from above (+) and below (−).

For the explicit forms given by equations (44) and (45), we have evaluated

$$\mathbf{U}(\xi) = \int_{-1}^{1} \mathbf{\Phi}(\xi, \mu)\, d\mu \tag{53}$$

to find

$$\mathbf{U}(\pm\nu_i) = \begin{bmatrix} -\Lambda_{12}(\nu_i) \\ \Lambda_{11}(\nu_i) \end{bmatrix}, \qquad \mathbf{U}^{(2)}(\nu) = \begin{bmatrix} -\Lambda_{12}(\nu) \\ \Lambda_{11}(\nu) \end{bmatrix}, \tag{54a,b}$$

$$\mathbf{U}_1^{(1)}(\nu) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{U}_2^{(1)}(\nu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{54c,d}$$

In regard to eventual applications, we would like to list the following inner-product integrals:

$$\int_0^1 \tilde{\mathbf{\Theta}}(\nu_j, \mu)\mathbf{\Phi}(-\nu_i, \mu)\mu\, d\mu = \frac{\nu_j\nu_i}{\nu_j + \nu_i}\, \tilde{\mathbf{U}}(\nu_j)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu_j)\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu_i)\mathbf{C}\mathbf{U}(\nu_i) \tag{55a}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}_\alpha^{(1)}(\nu, \mu)\mathbf{\Phi}(-\nu_i, \mu)\mu\, d\mu = \frac{\nu\nu_i}{\nu + \nu_i}\, \tilde{\mathbf{V}}_\alpha(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu_i)\mathbf{C}\mathbf{U}(\nu_i) \tag{55b}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}^{(2)}(\nu, \mu)\mathbf{\Phi}(-\nu_i, \mu)\mu\, d\mu = \frac{\nu\nu_i}{\nu + \nu_i}\, \tilde{\mathbf{U}}^{(2)}(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu_i)\mathbf{C}\mathbf{U}(\nu_i) \tag{55c}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}(\nu_i, \mu)\mathbf{\Phi}_\beta^{(1)}(-\nu, \mu)\mu\, d\mu = \frac{\nu\nu_i}{\nu + \nu_i}\, \tilde{\mathbf{U}}(\nu_i)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu_i)\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu)\mathbf{C}\mathbf{U}_\beta^{(1)}(\nu) \tag{55d}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}_\alpha^{(1)}(\nu', \mu)\mathbf{\Phi}_\beta^{(1)}(-\nu, \mu)\mu \, d\mu = \frac{\nu\nu'}{\nu + \nu'} \tilde{\mathbf{V}}_\alpha(\nu')\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu')\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu)\mathbf{C}\mathbf{U}_\beta^{(1)}(\nu) \tag{55e}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}^{(2)}(\nu', \mu)\mathbf{\Phi}_\beta^{(1)}(-\nu, \mu)\mu \, d\mu = \frac{\nu\nu'}{\nu + \nu'} \tilde{\mathbf{U}}^{(2)}(\nu')\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu')\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu)\mathbf{C}\mathbf{U}_\beta^{(1)}(\nu) \tag{55f}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}(\nu_i, \mu)\mathbf{\Phi}^{(2)}(-\nu, \mu)\mu \, d\mu = \frac{\nu\nu_i}{\nu + \nu_i} \tilde{\mathbf{U}}(\nu_i)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu_i)\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu)\mathbf{C}\mathbf{U}^{(2)}(\nu) \tag{55g}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}_\alpha^{(1)}(\nu', \mu)\mathbf{\Phi}^{(2)}(-\nu, \mu)\mu \, d\mu = \frac{\nu\nu'}{\nu + \nu'} \tilde{\mathbf{V}}_\alpha(\nu')\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu')\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu)\mathbf{C}\mathbf{U}^{(2)}(\nu) \tag{55h}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}^{(2)}(\nu', \mu)\mathbf{\Phi}^{(2)}(-\nu, \mu)\mu \, d\mu = \frac{\nu\nu'}{\nu + \nu'} \tilde{\mathbf{U}}^{(2)}(\nu')\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu')\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu)\mathbf{C}\mathbf{U}^{(2)}(\nu), \tag{55i}$$

where ν and ν' are both non-negative. In addition

$$\mathbf{V}_1(\nu) = N_{22}(\nu)\mathbf{U}_1^{(1)}(\nu) - N_{12}(\nu)\mathbf{U}_2^{(1)}(\nu) \tag{56a}$$

and

$$\mathbf{V}_2(\nu) = N_{11}(\nu)\mathbf{U}_2^{(1)}(\nu) - N_{21}(\nu)\mathbf{U}_1^{(1)}(\nu). \tag{56b}$$

Having proved the half-range orthogonality theorem and evaluated all required normalization integrals, we may express the solution to expansions of the form

$$\mathbf{\Phi}(\mu) = \sum_{i=1}^{\kappa} A(\nu_i)\mathbf{\Phi}(\nu_i, \mu) + \int_0^{1/\sigma} [A_1^{(1)}(\nu)\mathbf{\Phi}_1^{(1)}(\nu, \mu) + A_2^{(1)}(\nu)\mathbf{\Phi}_2^{(1)}(\nu, \mu)] \, d\nu + \int_{1/\sigma}^1 A^{(2)}(\nu)\mathbf{\Phi}^{(2)}(\nu, \mu) \, d\nu, \qquad \mu \in (0, 1), \tag{57}$$

as

$$A(\nu_i) = \frac{1}{N(\nu_i)} \int_0^1 \tilde{\mathbf{\Theta}}(\nu_i, \mu)\mathbf{\Phi}(\mu)\mu \, d\mu, \tag{58a}$$

$$A_\alpha^{(1)}(\nu) = \frac{1}{N^{(1)}(\nu)} \int_0^1 \tilde{\mathbf{\Theta}}_\alpha^{(1)}(\nu, \mu)\mathbf{\Phi}(\mu)\mu \, d\mu, \tag{58b}$$

and

$$A^{(2)}(\nu) = \frac{1}{N^{(2)}(\nu)} \int_0^1 \tilde{\mathbf{\Theta}}^{(2)}(\nu, \mu)\mathbf{\Phi}(\mu)\mu \, d\mu. \tag{58c}$$

We note that should the right-hand side of equation (57) be formally extended to negative μ, then the resulting left-hand side $\mathbf{\Phi}^*(-\mu)$, $\mu \in (0, 1)$, can be obtained from

$$\mathbf{\Phi}^*(-\mu) = \mathbf{\Gamma}\left(\frac{1}{\sigma} : \mu\right), \qquad \mu \in (0, 1), \tag{59a}$$

where

$$\mathbf{\Sigma}\mathbf{\Gamma}(\mu) = \frac{1}{2\mu} \int_0^1 \mathbf{S}(\mu, \mu')\mathbf{\Theta}(\mu')\mathbf{\Sigma}\mathbf{\Phi}(\sigma : \mu') \, d\mu', \qquad \mu \in (0, 1), \tag{59b}$$

with

$$\mathbf{S}(\mu, \mu') = \frac{2\mu\mu'}{\mu + \mu'} \mathbf{H}(\mu)\mathbf{C}\tilde{\mathbf{H}}(\mu'). \tag{59c}$$

Here we have introduced the notation

$$\mathbf{F}(\sigma\colon \mu) = \begin{bmatrix} f_1(\sigma\mu) \\ f_2(\mu) \end{bmatrix}, \tag{60}$$

where $f_1(\mu)$ and $f_2(\mu)$ are the elements of an arbitrary two-vector $\mathbf{F}(\mu)$.

Equations (59) are, of course, related to the work of PAHOR and SHULTIS (1969) which was based on CHANDRASEKHAR's (1950) invariance principles. We believe equations (59) are more convenient, however, since $\mathbf{S}(\mu, \mu')$ can be expressed simply in terms of the H-matrix, without the need for PAHOR and SHULTIS' (1969) special matrix product. Should no additional information be required, equations (59) may be used in the manner of CHANDRASEKHAR (1950) to deduce the exit distribution for typical half-space problems.

4. THE H-MATRIX

It is clear from the foregoing analysis that proper evaluation of the **H**-matrix is of primary importance to any half-space problems defined by the considered two-group transport equation. As we will demonstrate in Section 5, the solutions to typical half-space problems can be expressed concisely in terms of established quantities and the **H**-matrix. We would like therefore to report the procedure used here to compute the **H**-matrix and to list our numerical results for a few selected cases.

SIEWERT *et al.* (1972) proved the existence of a unique solution to the defining equations (28) for the considered cases and subsequently that the non-linear **H**-equation along with the linear constraints also uniquely specifies $\mathbf{H}(\mu)$. We prefer the non-linear form for computations, and thus we seek a numerical solution to

$$\mathbf{H}(\mu) = \mathbf{I} + \mu\mathbf{H}(\mu)\mathbf{C}\int_0^1 \widetilde{\mathbf{H}}(\mu')\mathbf{\Theta}(\mu')\frac{d\mu'}{\mu' + \mu}, \qquad \mu \in [0, 1], \tag{61a}$$

and

$$\left[\mathbf{I} + \nu_i\int_0^1 \widetilde{\mathbf{H}}(\mu')\mathbf{\Psi}(\mu')\frac{d\mu'}{\mu' - \nu_i}\right]\mathbf{M}(\nu_i) = \mathbf{0}, \qquad i = 1, 2, \ldots, \kappa. \tag{61b}$$

Since equation (61a) is not sufficient to specify $\mathbf{H}(\mu)$, $\mu \in [0, 1]$, we would like, in the manner of PAHOR (1968) and KRIESE and SIEWERT (1971), to develop a non-linear equation which in fact incorporates the constraints, equation (61b). For the sake of brevity, we report here our preferred method for computing $\mathbf{H}(\mu)$ only for the case $\kappa = 1$.

If we introduce

$$\mathbf{B}(z) = \begin{bmatrix} \dfrac{\nu_1(1 + z)}{\nu_1 + z} & 0 \\ 0 & 1 \end{bmatrix} \tag{62}$$

and

$$\mathbf{T} = \begin{bmatrix} c_{11}M_1(\nu_1) + c_{12}M_2(\nu_1) & -M_2(\nu_1) \\ c_{12}M_1(\nu_1) + c_{22}M_2(\nu_1) & M_1(\nu_1) \end{bmatrix}, \tag{63}$$

where the elements of $\mathbf{M}(\nu_1)$ are normalized such that $\det \mathbf{T} = 1$, then we can substitute

$$\mathbf{H}(z) = \mathbf{T}\mathbf{B}(z)\mathbf{L}(z)\mathbf{T}^{-1}, \qquad \kappa = 1, \tag{64}$$

into equations (61) to obtain

$$\mathbf{L}(\mu) = \mathbf{I} + \mu\mathbf{L}(\mu)\,\mathbf{\Delta}\int_0^1 \widetilde{\mathbf{L}}(\mu')\mathbf{R}(\mu')\frac{d\mu'}{\mu' + \mu}, \qquad \mu \in [0, 1]. \tag{65}$$

Here the matrix $\mathbf{R}(\mu)$ is given by

$$\mathbf{R}(\mu) = \mathbf{B}(\mu)\tilde{\mathbf{T}}\mathbf{\Theta}(\mu)\mathbf{T}\mathbf{B}(-\mu), \quad \text{and} \quad \mathbf{\Delta} = \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix}. \tag{66}$$

We note that equation (65) is sufficient to define $\mathbf{L}(\mu)$ and that the resulting $\mathbf{H}(\mu)$ computed from equation (64) will inherently satisfy the constraint, equation (61b), as well as equation (61a).

We have solved equation (65) iteratively by employing the improved Gaussian-quadrature scheme of Kronrod (1965) to represent the integration process. The computations were performed in double-precision arithmetic on an IBM 360/75 machine, and numerous checks were incorporated in the calculation. Since a discussion of the various checks used for a similar calculation of the H-matrix for the scattering of polarized light has recently been given (Schnatz and Siewert, 1971), we note only that we have used all of the analogous checking procedures here, and the indication of the accuracy of the computed H-matrix was, for the cases considered, comparable to that of Schnatz and Siewert (1971).

The **H**-matrix, of course, satisfies several identities that are useful for computational and/or analytical purposes. For example, we note that

$$[\mathbf{H}_o - \mathbf{I}]\mathbf{C}[\tilde{\mathbf{H}}_o - \mathbf{I}] = \mathbf{C}\mathbf{\Lambda}(\infty), \tag{67}$$

where

$$\tilde{\mathbf{H}}_0 = \int_0^1 \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\,\mathrm{d}\mu. \tag{68}$$

Also, the analytical results

$$H(z) = [\Lambda(\infty)]^{-(1/2)} \frac{1+z}{\nu_1 + z} \exp\left[-\frac{1}{\pi}\int_0^1 \arg \Lambda^+(\mu)\frac{\mathrm{d}\mu}{\mu + z}\right], \qquad \kappa = 1, \tag{69a}$$

and

$$H(z) = [\Lambda(\infty)]^{-(1/2)} \frac{(1+z)^2}{(\nu_1 + z)(\nu_2 + z)} \exp\left[-\frac{1}{\pi}\int_0^1 \arg \Lambda^+(\mu)\frac{\mathrm{d}\mu}{\mu + z}\right], \qquad \kappa = 2, \tag{69b}$$

are available for $H(z) = \det \mathbf{H}(z)$. We note further that $H(\mu)$ satisfies the non-linear equation

$$H(\mu) = 1 + \mu H(\mu)\int_0^1 H(\mu')\psi(\mu')\frac{\mathrm{d}\mu'}{\mu' + \mu}, \qquad \mu \in [0, 1], \tag{70a}$$

where the characteristic function is

$$\psi(\mu) = c_{11}\theta(\mu) + c_{22} - 2\mu C\left\{[\tau(\mu) + \tau(\sigma\mu)]\theta(\mu) + \tau\left(\frac{1}{\sigma\mu}\right)[1 - \theta(\mu)]\right\}, \tag{70b}$$

plus the linear constraints

$$1 + \nu_i\int_0^1 H(\mu')\psi(\mu')\frac{\mathrm{d}\mu'}{\mu' - \nu_i} = 0, \qquad i = 1, 2, \ldots, \kappa. \tag{70c}$$

For the purpose of reporting sample calculations of the **H**-matrix, we consider the data sets used by Metcalf and Zweifel (1968) to describe light-water media; the data sets are listed in Table 1. We note that the parameters required in equations

TABLE 1.—TWO-GROUP DATA SETS*

Case	σ_1	σ_2	σ_{11}	σ_{12}	σ_{21}	σ_{22}
I	4·8822	3·2343	3·8180	0·3524	1·0326	2·8669
II	4·9270	3·1686	3·7953	0·3239	1·0345	2·8005
III	5·0914	3·0707	3·7659	0·2705	1·0454	2·6828
IV	5·3220	2·9738	3·6906	0·2164	1·0481	2·5341

* METCALF and ZWEIFEL (1968).

TABLE 2.—THE H-MATRIX FOR SELECTED DATA SETS

Case	μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
	0·0	1·00000	0·0	0·0	1·00000
	0·1	1·19747	0·080405	0·059095	1·19430
	0·2	1·32793	0·15980	0·11189	1·34372
	0·3	1·43538	0·24050	0·16299	1·47926
	0·4	1·52868	0·32178	0·21280	1·60629
I	0·5	1·61198	0·40305	0·26146	1·72705
	0·6	1·68773	0·48388	0·30898	1·84276
	0·7	1·75749	0·56398	0·35542	1·95419
	0·8	1·82236	0·64312	0·40079	2·06183
	0·9	1·88315	0·72116	0·44512	2·16607
	1·0	1·94045	0·79801	0·48843	2·26718
	0·0	1·00000	0·0	0·0	1·00000
	0·1	1·19295	0·073071	0·053261	1·18675
	0·2	1·31650	0·14311	0·099096	1·32661
	0·3	1·41580	0·21273	0·14231	1·45087
	0·4	1·50009	0·28145	0·18348	1·56510
II	0·5	1·57375	0·34888	0·22286	1·67173
	0·6	1·63937	0·41477	0·26059	1·77215
	0·7	1·69863	0·47896	0·29679	1·86723
	0·8	1·75271	0·54136	0·33154	1·95763
	0·9	1·80247	0·60196	0·36494	2·04381
	1·0	1·84855	0·66074	0·39706	2·12617
	0·0	1·00000	0·0	0·0	1·00000
	0·1	1·18422	0·061846	0·043981	1·17519
	0·2	1·29573	0·11834	0·079423	1·30140
	0·3	1·38173	0·17255	0·11141	1·41029
	0·4	1·45212	0·22445	0·14079	1·50777
III	0·5	1·51164	0·27399	0·16800	1·59654
	0·6	1·56307	0·32118	0·19333	1·67823
	0·7	1·60822	0·36608	0·21700	1·75393
	0·8	1·64834	0·40879	0·23917	1·82442
	0·9	1·68434	0·44942	0·26001	1·89035
	1·0	1·71691	0·48807	0·27963	1·95219
	0·0	1·00000	0·0	0·0	1·00000
	0·1	1·17225	0·050886	0·035120	1·16236
	0·2	1·27010	0·094941	0·061357	1·27500
	0·3	1·34238	0·13569	0·083932	1·36955
	0·4	1·39945	0·17353	0·10390	1·45218
IV	0·5	1·44621	0·20869	0·12181	1·52585
	0·6	1·48550	0·24141	0·13802	1·59233
	0·7	1·51912	0·27188	0·15280	1·65283
	0·8	1·54833	0·30032	0·16634	1·70825
	0·9	1·57398	0·32690	0·17882	1·75929
	1·0	1·59675	0·35178	0·19035	1·80648

(1) and (3) can be obtained from the data sets of Table 1 in the following manner:

$$\sigma = \frac{\sigma_1}{\sigma_2}, \qquad q_{ij} = \frac{1}{2\sigma_2}\sigma_{ij} \quad \text{and} \quad \mathbf{C} = \mathbf{P}\mathbf{Q}\mathbf{P}^{-1}, \tag{71a,b,c}$$

where

$$\mathbf{P} = \begin{bmatrix} (q_{21}/q_{12})^{1/2} & 0 \\ 0 & 1 \end{bmatrix}. \tag{72}$$

The data sets of Table 1 have been used to compute the corresponding **C**-matrices, and our numerical values of the resulting **H**-matrix are listed in Table 2. As previously mentioned, we believe the various checking procedures we have used are sufficient to suggest that the results given in Table 2 are accurate to within the usual round-off convention.

5. HALF-SPACE APPLICATIONS

Having established the necessary analytical formalism and having computed the required **H**-matrix, we would now like to solve the typical half-space problems. Since the data sets listed in Table 1 all correspond to $\kappa = 1$ and $\Lambda(\infty) > 0$, we will abbreviate our solutions here to reflect these conditions. More discussion of the case $\kappa = 2$ will be given elsewhere (ISHIGURO, 1972).

For the Milne problem, we seek a diverging (as $x \to \infty$) solution to

$$\mu \frac{\partial}{\partial x}\mathbf{I}(x, \mu) + \mathbf{\Sigma}\mathbf{I}(x, \mu) = \mathbf{Q}\int_{-1}^{1}\mathbf{I}(x, \mu')\,\mathrm{d}\mu' \tag{73}$$

subject to the boundary conditions

$$\lim_{x\to\infty}\mathbf{I}(x, \mu)\mathrm{e}^{-x} = \mathbf{0}, \quad \text{and} \quad \mathbf{I}(0, \mu) = \mathbf{0}, \qquad \mu \in (0, 1). \tag{74a,b}$$

We write the solution $\mathbf{I}_M(x, \mu)$ as

$$\mathbf{I}_M(x, \mu) = \mathbf{P}^{-1}\mathbf{\Psi}_M(x, \mu), \tag{75}$$

where $\mathbf{\Psi}_M(x, \mu)$ is the diverging (as $x \to \infty$) solution of equation (3);

$$\lim_{x\to\infty}\mathbf{\Psi}_M(x, \mu)\mathrm{e}^{-x} = \mathbf{0} \quad \text{and} \quad \mathbf{\Psi}_M(0, \mu) = \mathbf{0}, \qquad \mu \in (0, 1). \tag{76a,b}$$

It follows that

$$\begin{aligned}\mathbf{\Psi}_M(x, \mu) = {} & A(\nu_1)\mathbf{\Phi}(\nu_1, \mu)\mathrm{e}^{-x/\nu_1} + \mathbf{\Phi}(-\nu_1, \mu)\mathrm{e}^{x/\nu_1} \\ & + \int_0^{1/\sigma}[A_1^{(1)}(\nu)\mathbf{\Phi}_1^{(1)}(\nu, \mu) + A_2^{(1)}(\nu)\mathbf{\Phi}_2^{(1)}(\nu, \mu)]\mathrm{e}^{-x/\nu}\,\mathrm{d}\nu \\ & + \int_{1/\sigma}^{1} A^{(2)}(\nu)\mathbf{\Phi}^{(2)}(\nu, \mu)\mathrm{e}^{-x/\nu}\,\mathrm{d}\nu, \qquad \kappa = 1,\end{aligned} \tag{77}$$

where the arbitrary normalization $A(-\nu_1) = 1$ has been imposed. The eigenvectors $\mathbf{\Phi}(\pm\nu_1, \mu)$, $\mathbf{\Phi}_1^{(1)}(\nu, \mu)$, $\mathbf{\Phi}_2^{(1)}(\nu, \mu)$ and $\mathbf{\Phi}^{(2)}(\nu, \mu)$, of course, are given explicitly by equations (44) and (45). The solution given by equation (77) clearly satisfies equation (3) and the condition at infinity. Constraining equation (77) to meet the free-surface condition, equation (76b), yields

$$-\mathbf{\Phi}(-\nu_1, \mu) = A(\nu_1)\mathbf{\Phi}(\nu_1, \mu) + \int_0^{1/\sigma} [A_1^{(1)}(\nu)\mathbf{\Phi}_1^{(1)}(\nu, \mu) + A_2^{(1)}(\nu)\mathbf{\Phi}_2^{(1)}(\nu, \mu)]\, d\nu + \int_{1/\sigma}^1 A^{(2)}(\nu)\mathbf{\Phi}^{(2)}(\nu, \mu)\, d\nu, \qquad \mu \in (0, 1). \quad (78)$$

The half-range expansion theory discussed by SIEWERT *et al.* (1972) ensures that equation (78) is solvable. We therefore multiply equation (78) by $\mu\tilde{\mathbf{\Theta}}(\nu_1, \mu)$ and integrate over μ from 0–1 to find, after invoking our half-range orthogonality theorem and equations (48a) and (55a),

$$A(\nu_1) = -\tfrac{1}{2}\nu_1 \frac{1}{N(\nu_1)} \tilde{\mathbf{U}}(\nu_1)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu_1)\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu_1)\mathbf{C}\mathbf{U}(\nu_1). \quad (79a)$$

In a similar manner, we find

$$A_\alpha^{(1)}(\nu) = -\frac{\nu\nu_1}{\nu + \nu_1} \frac{1}{N^{(1)}(\nu)} \tilde{\mathbf{V}}_\alpha(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu_1)\mathbf{C}\mathbf{U}(\nu_1), \qquad \nu \in \left(0, \frac{1}{\sigma}\right), \quad \alpha = 1, 2, \quad (79b)$$

and

$$A^{(2)}(\nu) = -\frac{\nu\nu_1}{\nu + \nu_1} \frac{1}{N^{(2)}(\nu)} \tilde{\mathbf{U}}^{(2)}(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\mathbf{C}^{-1}\mathbf{H}^{-1}(\nu_1)\mathbf{C}\mathbf{U}(\nu_1), \qquad \nu \in \left(\frac{1}{\sigma}, 1\right), \quad (79c)$$

which completes the solution.

For $x = 0$, equation (77) can be simplified, or the S-matrix along with equations (59) and (78) can be used, to yield the exit distribution for the Milne problem. We find

$$\mathbf{\Psi}_M(0, -\mu) = \mathbf{E}(\nu_1, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\nu_1)\mathbf{C}\mathbf{U}(\nu_1), \quad (80a)$$

or, after equation (75) is invoked,

$$\mathbf{I}_M(0, -\mu) = \mathbf{P}^{-1}\mathbf{E}(\nu_1, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\nu_1)\mathbf{C}\mathbf{U}(\nu_1). \quad (80b)$$

The Milne-problem extrapolated end-point is defined by

$$\int_{-1}^1 \mathbf{I}_{Masy}(-z_0, \mu)\, d\mu = \mathbf{P}^{-1}\int_{-1}^1 \mathbf{\Psi}_{Masy}(-z_0, \mu)\, d\mu = \mathbf{0}, \quad (81)$$

where $\mathbf{\Psi}_{Masy}(x, \mu)$ is the asymptotic solution obtained by neglecting the continuum contribution to $\mathbf{\Psi}_M(x, \mu)$ in equation (77). We find

$$z_0 = -\tfrac{1}{2}\nu_1 \ln[-A(\nu_1)], \quad (82)$$

where $A(\nu_1)$ is given by equation (79a). In Table 3 we list our calculated values of the discrete eigenvalue ν_1 and the Milne-problem extrapolated end-point, along with those obtained by METCALF and ZWEIFEL (1968) by solving numerically the system of singular-integral equations defining the Milne problem. Our results were obtained directly from the non-linear, but regular, H-equation. In Table 4 we report the exit distribution for the Milne problem, as given by equation (80b). We note that our

TABLE 3.—THE DISCRETE EIGENVALUE AND THE MILNE-PROBLEM EXTRAPOLATED END-POINT

Case	ν_1	ν_1*	z_0	z_0*
I	7·190978	7·1869	0·665826	0·6658
II	4·179546	4·1793	0·678359	0·6783
III	2·595565	2·5958	0·712132	0·7121
IV	1·936041	1·9361	0·761386	0·7613

* METCALF and ZWEIFEL (1968).

entries in Tables 3 and 4 are in exact agreement with REITH (1971) who used a method based on a scaler, but not very tractable, Fredholm equation.

For the half-space albedo problem we seek a bounded (as $x \to \infty$) solution to equation (3) such that

$$\mathbf{\Psi}(0, \mu) = \delta(\mu - \mu_0)\mathbf{F}, \qquad \mu, \mu_0 \in (0, 1), \tag{83}$$

where F is a given constant vector. The solution $\mathbf{\Psi}_a(x, \mu)$ can be written as

$$\mathbf{\Psi}_a(x, \mu) = A(\nu_1)\mathbf{\Phi}(\nu_1, \mu)e^{-x/\nu_1} + \int_0^{1/\sigma} [A_1^{(1)}(\nu)\mathbf{\Phi}_1^{(1)}(\nu, \mu)$$

$$+ A_2^{(1)}(\nu)\mathbf{\Phi}_2^{(1)}(\nu, \mu)]e^{-x/\nu}\, d\nu + \int_{1/\sigma}^{1} A^{(2)}(\nu)\mathbf{\Phi}^{(2)}(\nu, \mu)e^{-x/\nu}\, d\nu, \qquad \kappa = 1, \tag{84}$$

TABLE 4.—THE MILNE-PROBLEM EXIT DISTRIBUTION $\mathbf{I}_M(0, -\mu)$

μ	$I_1(0, -\mu)$ Case I	Case II	Case III	Case IV
0·05	0·004841	0·007121	0·008512	0·008193
0·10	0·005277	0·007755	0·009249	0·008873
0·20	0·006066	0·008905	0·010587	0·010106
0·30	0·006803	0·009982	0·011848	0·011273
0·40	0·007515	0·011026	0·013079	0·012419
0·50	0·008210	0·012051	0·014300	0·013568
0·60	0·008896	0·013068	0·015526	0·014736
0·70	0·009574	0·014082	0·016767	0·015935
0·80	0·010249	0·015098	0·018032	0·017177
0·90	0·010921	0·016121	0·019329	0·018473
1·00	0·011592	0·017152	0·020664	0·019835

μ	$I_2(0, -\mu)$ Case I	Case II	Case III	Case IV
0·05	0·017021	0·028432	0·045837	0·064905
0·10	0·018736	0·031288	0·050399	0·071267
0·20	0·021900	0·036585	0·058965	0·083413
0·30	0·024920	0·041686	0·067391	0·095692
0·40	0·027879	0·046740	0·075964	0·108614
0·50	0·030813	0·051818	0·084855	0·122564
0·60	0·033739	0·056965	0·094205	0·137934
0·70	0·036672	0·062216	0·104154	0·155190
0·80	0·039620	0·067601	0·114855	0·174927
0·90	0·042590	0·073151	0·126485	0·197947
1·00	0·045588	0·078894	0·139257	0·225375

where the expansion coefficients must satisfy

$$\delta(\mu-\mu_0)\mathbf{F} = A(\nu_1)\boldsymbol{\Phi}(\nu_1,\mu) + \int_0^{1/\sigma} [A_1^{(1)}(\nu)\boldsymbol{\Phi}_1^{(1)}(\nu,\mu) + A_2^{(1)}(\nu)\boldsymbol{\Phi}_2^{(1)}(\nu,\mu)]\,\mathrm{d}\nu + \int_{1/\sigma}^1 A^{(2)}(\nu)\boldsymbol{\Phi}^{(2)}(\nu,\mu)\,\mathrm{d}\nu, \qquad \mu,\mu_0 \in (0,1). \quad (85)$$

If we multiply equation (85) by $\mu\tilde{\boldsymbol{\Theta}}(\nu_1, \mu)$ and integrate over μ from 0–1, we find

$$A(\nu_1) = \frac{\mu_0}{N(\nu_1)} \tilde{\mathbf{U}}(\nu_1)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu_1)\tilde{\mathbf{h}}(\mu_0)\mathbf{E}(\nu_1,\mu_0)\mathbf{F}. \quad (86)$$

In a similar manner, we find the continuum expansion coefficients to be

$$A_\alpha^{(1)}(\nu) = \frac{\mu_0}{N^{(1)}(\nu)} \tilde{\mathbf{V}}_\alpha(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\tilde{\mathbf{h}}(\mu_0)\mathbf{E}(\nu,\mu_0)\mathbf{F}, \qquad \nu \in \left(0, \frac{1}{\sigma}\right), \qquad \alpha = 1, 2, \quad (87)$$

and

$$A^{(2)}(\nu) = \frac{\mu_0}{N^{(2)}(\nu)} \tilde{\mathbf{U}}^{(2)}(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\tilde{\mathbf{h}}(\mu_0)\mathbf{E}(\nu,\mu_0)\mathbf{F}, \qquad \nu \in \left(\frac{1}{\sigma}, 1\right), \quad (88)$$

which completes the solution, which is restricted, of course, to non-multiplying media.

We now consider a constant-source problem and thus wish to construct $\boldsymbol{\Psi}_s(x, \mu)$, a solution (bounded as $x \to \infty$) to

$$\mu\frac{\partial}{\partial x}\boldsymbol{\Psi}(x,\mu) + \boldsymbol{\Sigma}\boldsymbol{\Psi}(x,\mu) = \mathbf{C}\int_{-1}^1 \boldsymbol{\Psi}(x,\mu')\,\mathrm{d}\mu' + \mathbf{S}, \quad (89)$$

where S is a given constant, subject to

$$\boldsymbol{\Psi}_s(0,\mu) = \mathbf{0}, \qquad \mu \in (0, 1). \quad (90)$$

We consider here non-conservative media, $\Lambda(\infty) > 0$, and thus we write

$$\boldsymbol{\Psi}_s(x,\mu) = A(\nu_1)\boldsymbol{\Phi}(\nu_1,\mu)\mathrm{e}^{-x/\nu_1} + \int_0^{1/\sigma} [A_1^{(1)}(\nu)\boldsymbol{\Phi}_1^{(1)}(\nu,\mu) + A_2^{(1)}(\nu)\boldsymbol{\Phi}_2^{(1)}(\nu,\mu)]\mathrm{e}^{-x/\nu}\,\mathrm{d}\nu + \int_{1/\sigma}^1 A^{(2)}(\nu)\boldsymbol{\Phi}^{(2)}(\nu,\mu)\mathrm{e}^{-x/\nu}\,\mathrm{d}\nu + \boldsymbol{\Psi}_p(x,\mu), \qquad \kappa = 1, \quad (91)$$

where

$$\boldsymbol{\Psi}_p(x,\mu) = [\boldsymbol{\Sigma} - 2\mathbf{C}]^{-1}\mathbf{S} \quad (92)$$

is the required particular solution to equation (89). Entering equation (91) into equation (90), we obtain

$$-\boldsymbol{\Psi}_p(0,\mu) = A(\nu_1)\boldsymbol{\Phi}(\nu_1,\mu) + \int_0^{1/\sigma} [A_1^{(1)}(\nu)\boldsymbol{\Phi}_1^{(1)}(\nu,\mu) + A_2^{(1)}(\nu)\boldsymbol{\Phi}_2^{(1)}(\nu,\mu)]\,\mathrm{d}\nu + \int_{1/\sigma}^1 A^{(2)}(\nu)\boldsymbol{\Phi}^{(2)}(\nu,\mu)\,\mathrm{d}\nu, \quad (93)$$

and thus for this case we find

$$A(\nu_1) = \nu_1\frac{1}{N(\nu_1)} \tilde{\mathbf{U}}(\nu_1)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu_1)\mathbf{C}^{-1}(\mathbf{H}_0 - \mathbf{I})^{-1}\mathbf{S}, \quad (94)$$

$$A_\alpha^{(1)}(\nu) = \nu\frac{1}{N^{(1)}(\nu)} \tilde{\mathbf{V}}_\alpha(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\mathbf{C}^{-1}(\mathbf{H}_0 - \mathbf{I})^{-1}\mathbf{S}, \quad (95)$$

and

$$A^{(2)}(\nu) = \nu \frac{1}{N^{(2)}(\nu)} \tilde{\mathbf{U}}^{(2)}(\nu)\mathbf{C}\tilde{\mathbf{H}}^{-1}(\nu)\mathbf{C}^{-1}(\mathbf{H}_0 - \mathbf{I})^{-1}\mathbf{S}. \tag{96}$$

Since the expansion coefficients for all considered cases have been expressed in terms of the calculated **H**-matrix, further numerical results, though available (ISHIGURO, 1972), are not given here.

Acknowledgements—The authors wish to express their appreciation to E. E. BURNISTON and J. T. KRIESE for many helpful discussions regarding this paper. This work was supported in part by the National Science Foundation through grant GK 11935.

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