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TWO-GROUP NEUTRON-TRANSPORT THEORY: EXISTENCE AND UNIQUENESS OF THE H-MATRIX

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Abstract—The system of singular-integral equations and linear constraints which define mathematically the H-matrix relevant to a class of two-group neutron-transport problems is shown to possess a unique solution. The analysis is based on a matrix version of the Riemann problem appropriate to which a discussion of a class of canonical solutions is given.

1. INTRODUCTION

WE ARE concerned here with the steady-state, one-dimensional, two-group neutron-transport equation written conveniently as

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Sigma}\mathbf{\Psi}(x, \mu) = \mathbf{C}\int_{-1}^{1} \mathbf{\Psi}(x, \mu')\, d\mu', \tag{1}$$

where the elements of the two-vector $\mathbf{\Psi}(x, \mu)$ are the group angular fluxes, μ is the directional cosine of the propagating radiation (as measured from the positive x-axis) and x is the optical variable defined in terms of the smaller of the two total cross-sections. With the optical variable so defined, the $\mathbf{\Sigma}$-matrix is given by

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma > 1, \tag{2}$$

where σ is the ratio of the two total cross-sections. In addition, the transfer matrix, with arbitrary real positive elements c_{ij}, for isotropic scattering is denoted by $\mathbf{C}$. We shall, however, without loss of generality consider $\mathbf{C}$ to be symmetric since a general two-group model with isotropic scattering can be transformed by elementary methods to a form with a symmetric C-matrix.

We note that a general solution to equation (1) may be written as

$$\begin{aligned} \mathbf{\Psi}(x, \mu) = \sum_{i=1}^{\kappa} [A(\nu_i)\mathbf{\Phi}(\nu_i, \mu)e^{-x/\nu_i} + A(-\nu_i)\mathbf{\Phi}(-\nu_i, \mu)e^{x/\nu_i}] \\ + \int_{①} [A_1^{(1)}(\nu)\mathbf{\Phi}_1^{(1)}(\nu, \mu)e^{-x/\nu} + A_2^{(1)}(\nu)\mathbf{\Phi}_2^{(1)}(\nu, \mu)e^{-x/\nu}]\, d\nu \\ + \int_{②} A^{(2)}(\nu)\mathbf{\Phi}^{(2)}(\nu, \mu)e^{-x/\nu}\, d\nu, \end{aligned} \tag{3}$$

where $A(\pm\nu_i)$, $A_\alpha^{(1)}(\nu)$ and $A^{(2)}(\nu)$ are the expansion coefficients to be determined once the boundary conditions, subject to which the solution is to be constrained, are specified. In order to display the elementary solutions used in equation (3), we note (SIEWERT and ZWEIFEL, 1966) that $\nu \in ① \Rightarrow \nu \in (-(1/\sigma), 1/\sigma))$ and that

$$\mathbf{\Phi}_1^{(1)}(\nu, \mu) = \begin{bmatrix} c_{11}\nu P\nu\left(\frac{1}{\sigma\nu - \mu}\right) + \delta(\sigma\nu - \mu)[1 - 2c_{11}\nu \tanh^{-1} \sigma\nu] \\ c_{21}\nu P\nu\left(\frac{1}{\nu - \mu}\right) + \delta(\nu - \mu)[-2c_{21}\nu \tanh^{-1} \nu] \end{bmatrix} \tag{4a}$$

and

$$\mathbf{\Phi}_2^{(1)}(\nu,\mu)=\begin{bmatrix} c_{12}\nu Pv\left(\dfrac{1}{\sigma\nu-\mu}\right)+\delta(\sigma\nu-\mu)[-2c_{12}\nu\tanh^{-1}\sigma\nu] \\ c_{22}\nu Pv\left(\dfrac{1}{\nu-\mu}\right)+\delta(\nu-\mu)[1-2c_{22}\nu\tanh^{-1}\nu] \end{bmatrix}, \tag{4b}$$

whereas $\nu\in②\Rightarrow\nu\in(-1,-1/\sigma)U(1/\sigma,1)$, and thus

$$\mathbf{\Phi}^{(2)}(\nu,\mu)=\begin{bmatrix} \dfrac{c_{12}\nu}{\sigma\nu-\mu} \\ \nu f(\nu)Pv\left(\dfrac{1}{\nu-\mu}\right)+\lambda(\nu)\,\delta(\nu-\mu) \end{bmatrix}, \tag{5}$$

where

$$f(\nu)=c_{22}-2C\nu\tanh^{-1}\left(\frac{1}{\sigma\nu}\right) \tag{6a}$$

and

$$\lambda(\nu)=1-2c_{22}\nu\tanh^{-1}\nu-2c_{11}\nu\tanh^{-1}\left(\frac{1}{\sigma\nu}\right)+4C\nu^2\tanh^{-1}\nu\tanh^{-1}\left(\frac{1}{\sigma\nu}\right). \tag{6b}$$

Here we have introduced the notation $C=\det\mathbf{C}$. In addition, $Pv(1/x)$ and $\delta(x)$ denote the Cauchy principal-value distribution and the Dirac delta distribution, respectively.

The discrete eigenvectors are given by

$$\mathbf{\Phi}(\pm\nu_i,\mu)=\begin{bmatrix} \dfrac{c_{12}\nu_i}{\sigma\nu_i\mp\mu} \\ \dfrac{f(\nu_i)\nu_i}{\nu_i\mp\mu} \end{bmatrix}, \tag{7}$$

where the discrete eigenvalues $\pm\nu_i$, $i=1,2,\ldots,\kappa$, are the zeros in the cut plane of the dispersion function

$$\Lambda(z)=\det\mathbf{\Lambda}(z), \tag{8}$$

where

$$\mathbf{\Lambda}(z)=\mathbf{I}+z\int_{-1}^{1}\mathbf{\Psi}(\mu)\frac{d\mu}{\mu-z}, \tag{9}$$

with $\mathbf{I}$ denoting the unit matrix and the characteristic matrix $\mathbf{\Psi}(\mu)$, not to be confused with the angular flux $\mathbf{\Psi}(x,\mu)$, given by

$$\mathbf{\Psi}(\mu)=\mathbf{\Theta}(\mu)\mathbf{C}, \tag{10}$$

where

$$\mathbf{\Theta}(\mu)=\begin{bmatrix}\theta(\mu) & 0\\ 0 & 1\end{bmatrix},\quad \theta(\mu)=1,\quad \mu\in\left(-\frac{1}{\sigma},\frac{1}{\sigma}\right);\quad \theta(\mu)=0,\ \text{otherwise}. \tag{11}$$

SIEWERT and SHIEH (1967) have summarized the number and types of zeros of the dispersion function; we note that equation (8) has either one pair $\pm\nu_1$ or two pairs $\pm\nu_1$ and $\pm\nu_2$ of zeros in the complex plane cut from -1 to 1 along the real axis. Without loss of generality, we consider the real (imaginary) part of ν_α to be positive.

All of the formalism necessary for problems defined by full-range boundary conditions has been reported (SIEWERT and SHIEH, 1967). However for the considerably more meaningful problems defined by half-range boundary conditions, only the special case $\det \mathbf{C} = 0$ has been resolved definitively (SIEWERT and ZWEIFEL, 1966). Of course there are two simple special cases, *viz.* $\sigma = 1$ or an upper or lower triangular **C**-matrix, which can be solved analytically. In this and a related paper (SIEWERT and ISHIGURO, 1972) we develop the theory required to ensure that the extension of CASE's (1960) method of singular eigenfunction expansions yields rigorous results for more general ($\det \mathbf{C} \neq 0$) two-group models.

Basic to the use of the elementary solutions to equation (1) to solve half-space problems in two-group transport theory is the so-called half-range completeness theorem. Alternatively, some authors (e.g. PAHOR, 1968; PAHOR and SHULTIS, 1969) have made use of the invariance principles developed by CHANDRASEKHAR (1950) to express the solutions to half-space problems in terms of an S-matrix, which in turn can be expressed in terms of an **H**-matrix. Although the use of CHANDRASEKHAR's (1950) invariance principles or the method proposed by BOWDEN and MCCROSSON (1971) appears at first to avoid the difficulties normally encountered in proving a half-range expansion (or completeness) theorem, closer examination of these two methods reveals that this is not the case. In fact, to prove the existence of a unique solution to the equations defining the **H**-matrix is equivalent to demonstrating that the eigenvectors are half-range complete.

We proceed therefore to prove that a unique solution of the equations which define the **H**-matrix appropriate to equation (1) exists. The crux of our existence proof is an argument that a certain Riemann problem yields a canonical matrix with non-negative partial indices. The importance of these partial indices, with regard to half-range completeness theorems in multi-group theory, has recently been reviewed by BURNISTON, SIEWERT, SILVENNOINEN and ZWEIFEL (1971) and will be apparent in our discussions. The method used here to prove that the partial indices are non-negative requires proof that a certain polynomial $P_{11}(z)$ is not identically zero. We give in Appendix A definitive proof that $P_{11}(z) \neq 0$ for all cases for which $\det \mathbf{C} > 0$ and for those cases of $\det \mathbf{C} < 0$ for which $\pm\nu_1$ are imaginary $[\Lambda(\infty) < 0]$; we note that for $\det \mathbf{C} < 0$, equation (8) has only two zeros $\pm\nu_1$. The remaining case, since the theory for $\det \mathbf{C} = 0$ has been established (SIEWERT and ZWEIFEL, 1966), of $\det \mathbf{C} < 0$ with $\pm\nu_1$ real $[\Lambda(\infty) > 0]$ has not been satisfactorily resolved in general; however the analysis given in Appendix A is sufficient to show that at least a subset of these elusive cases can be included in our existence and uniqueness theorem.

2. BASIC ANALYSIS

Since a recent paper (SIEWERT and ISHIGURO, 1972) is devoted entirely to the use of the **H**-matrix for solving typical half-space problems in two-group neutron-transport theory, we focus our attention here on the required existence and uniqueness theorem.

We are able to prove the following result:

Theorem. *The equations*

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\nu)\boldsymbol{\Psi}(\nu)\frac{d\nu}{\nu-\mu}, \qquad \mu \in \left(0, \frac{1}{\sigma}\right), \tag{12a}$$

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu)\mathbf{M}^{(2)}(\mu) = \left[\mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\nu)\boldsymbol{\Psi}(\nu)\frac{d\nu}{\nu-\mu}\right]\mathbf{M}^{(2)}(\mu), \qquad \mu \in \left(\frac{1}{\sigma}, 1\right), \tag{12b}$$

and

$$\left[\mathbf{I} + \nu_i\int_0^1 \tilde{\mathbf{H}}(\nu)\boldsymbol{\Psi}(\nu)\frac{d\nu}{\nu-\nu_i}\right]\mathbf{M}(\nu_i) = \mathbf{0}, \qquad i = 1, 2, \ldots, \kappa, \qquad \kappa = 1 \text{ or } 2, \tag{12c}$$

possess a unique solution in the class of continuous functions $\mathscr{C}[0, 1]$ *for which* $\det \mathbf{C} > 0$ *and for all* $\det \mathbf{C} < 0$ *cases for which* $\Lambda(\infty) \leq 0$.

In equations (12) we have used the superscript tilde to denote the transpose operation and have introduced the quantities

$$\mathbf{M}^{(2)}(\mu) = \int_{-1}^1 \boldsymbol{\Phi}^{(2)}(\mu, \mu')\, d\mu' \quad \text{and} \quad \mathbf{M}(\nu_i) = \int_{-1}^1 \boldsymbol{\Phi}(\nu_i, \mu')\, d\mu', \tag{13}$$

which can be evaluated immediately from equations (5) and (7). In addition note that as z approaches, in a non-tangential manner, the branch cut $(-1, 1)$ from above $(+)$ and below $(-)$ the resulting boundary values of the dispersion matrix $\boldsymbol{\Lambda}(z)$ satisfy

$$\boldsymbol{\Lambda}^+(\mu) + \boldsymbol{\Lambda}^-(\mu) = 2\boldsymbol{\lambda}(\mu) \tag{14a}$$

and

$$\boldsymbol{\Lambda}^+(\mu) - \boldsymbol{\Lambda}^-(\mu) = 2\pi i\mu\boldsymbol{\Psi}(\mu), \tag{14b}$$

where

$$\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P\int_{-1}^1 \boldsymbol{\Psi}(\nu)\frac{d\nu}{\nu-\mu}. \tag{15}$$

In addition,

$$\Lambda(\infty) = 1 - \frac{2}{\sigma}c_{11} - 2c_{22} + \frac{4}{\sigma}C. \tag{16}$$

We note that equations (12) are quite similar to the equations which define the H-matrix for a problem in the scattering of polarized light (SIEWERT and BURNISTON, 1972); however, the fact that here the characteristic matrix $\boldsymbol{\Psi}(\mu)$ is singular for $\mu \in (1/\sigma, 1)$ does require special attention. Since $\boldsymbol{\Psi}(\mu)$ is singular for $\mu \in (1/\sigma, 1)$ we conclude from equations (12) that all of the elements of $\mathbf{H}(\mu)$ for $\mu \in [0, 1]$ are not involved in these equations. In fact, the presence of $\mathbf{M}^{(2)}(\mu)$ in equation (12b) is a consequence of $\boldsymbol{\Psi}(\mu)$ being singular for $\mu \in (1/\sigma, 1)$. If we denote the elements of $\mathbf{H}(\mu)$ by $H_{ij}(\mu)$, we observe that only $H_{11}(\mu)$ and $H_{12}(\mu)$, $\mu \in (0, 1/\sigma)$, and $H_{21}(\mu)$ and $H_{22}(\mu)$, $\mu \in (0, 1)$, are required in equations (12), and thus our theorem concerns only $\boldsymbol{\Theta}(\mu)\mathbf{H}(\mu)$, $\mu \in (0, 1)$.

Though, for the sake of brevity, we will not give an explicit derivation of equations

(12) here, we would like to make several comments on these defining equations. The technique reported by PAHOR (1968) and PAHOR and SHULTIS (1969), though restricted to non-multiplying media and inherently based on the proposition that the non-diverging normal modes are half-range complete (as discussed in our Appendix B), can be used to derive equations (12) for non-multiplying media. Our derivation of equations (12) may be viewed in a manner which allows for an obvious extension to the case of multiplying media. We find that equations (12) are sufficient conditions for the half-range basis set to be orthogonal, as described by SIEWERT and ISHIGURO (1972). Since this property of the **H**-matrix is useful for finite-slab problems as well as half-space problems, it is clearly desirable for $\mathbf{H}(\mu)$ to be defined by equations (12) for both multiplying and non-multiplying media.

To prove the theorem we make use of the equivalence of the given singular-integral equations to a certain matrix version of the classical Riemann problem. Therefore, in the manner of MUSKHELISHVILI (1953), introduce the sectionally analytic matrix

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_0^1 \tilde{\mathbf{H}}(\nu)\mathbf{\Psi}(\nu)\frac{d\nu}{\nu - z}, \tag{17}$$

which vanishes at least as fast as $1/z$ as $|z|$ tends to infinity. Invoking the Plemelj formulae (MUSKHELISHVILI, 1953), we note that the boundary values of $\mathbf{N}(z)$ satisfy

$$\pi i[\mathbf{N}^+(\mu) + \mathbf{N}^-(\mu)] = P\int_0^1 \tilde{\mathbf{H}}(\nu)\mathbf{\Psi}(\nu)\frac{d\nu}{\nu - \mu} \tag{18}$$

and

$$\mathbf{N}^+(\mu) - \mathbf{N}^-(\mu) = \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu). \tag{19}$$

If we now make use of equations (14) and (17) we find equations (12a) and (12b) can be reduced to the equivalent inhomogeneous Riemann problem

$$\tilde{\mathbf{N}}^+(\mu) = \mathbf{G}(\mu)\tilde{\mathbf{N}}^-(\mu) + \tilde{\mathbf{\Psi}}(\mu)[\tilde{\mathbf{\Lambda}}^-(\mu)]^{-1}, \qquad \mu \in (0, 1), \tag{20}$$

where

$$\mathbf{G}(\mu) = \tilde{\mathbf{\Lambda}}^+(\mu)[\tilde{\mathbf{\Lambda}}^-(\mu)]^{-1}. \tag{21}$$

Even though $\mathbf{\Lambda}^+(\mu)$ and $\mathbf{\Lambda}^-(\mu)$ are unbounded at $\mu = 1/\sigma$ and $\mu = 1$, the **G**-matrix can be defined to be continuous (though not Hölder continuous) for $\mu \in [0, 1]$. With $\mathbf{G}(0) = \mathbf{G}(1) = \mathbf{I}$, the analysis of MANDŽAVIDZE and HVEDELIDZE (1958) is sufficient to ensure the existence of a canonical solution $\mathbf{\Phi}(z)$ to the homogeneous problem

$$\mathbf{\Phi}^+(\mu) = \mathbf{G}(\mu)\mathbf{\Phi}^-(\mu), \qquad \mu \in [0, 1]. \tag{22}$$

Equation (20) thus can be solved to yield

$$\tilde{\mathbf{N}}(z) = \frac{1}{2\pi i}\mathbf{\Phi}(z)\left[\int_0^1 \mathbf{K}(\nu)\frac{d\nu}{\nu - z} + \mathbf{\vartheta}(z)\right], \tag{23}$$

where $\mathbf{\vartheta}(z)$ is a matrix of polynomials and

$$\mathbf{K}(\nu) = [\mathbf{\Phi}^+(\nu)]^{-1}\tilde{\mathbf{\Psi}}(\nu)[\tilde{\mathbf{\Lambda}}^-(\nu)]^{-1}. \tag{24}$$

The matrix $\mathbf{N}(z)$ given by equation (23) is a solution of equation (20) but is not unique. We now want to show that imposing the condition that $z\mathbf{N}(z)$ is bounded as $|z|$ tends to infinity so limits the number of arbitrary constants in $\boldsymbol{\vartheta}(z)$ that they are uniquely determined by equation (12c). Our approach will be to analyze the behavior of the canonical solution as $|z|$ tends to infinity. This then introduces the so-called partial indices κ_1 and κ_2 and the total index $\kappa = \kappa_1 + \kappa_2$. We note that the total index κ for the Riemann problem defined by equation (22) is either $\kappa = 1$ or $\kappa = 2$ depending on σ and $\mathbf{C}$ (SIEWERT and SHIEH, 1967).

Now since the matrix

$$\boldsymbol{\pi}(z) = \mathbf{C}\boldsymbol{\Lambda}(z)\bar{\boldsymbol{\Phi}}^{-1}(-z), \tag{25}$$

analytic in the complex plane cut from zero to one along the real axis, is also a solution to equation (22), we can write (MANDŽAVIDZE and HVEDELIDZE, 1958)

$$\boldsymbol{\pi}(z) = \boldsymbol{\Phi}(z)\mathbf{P}(z), \tag{26}$$

where $\mathbf{P}(z)$ is a matrix of polynomials. The factorization of the dispersion matrix is therefore established:

$$\mathbf{C}\boldsymbol{\Lambda}(z) = \boldsymbol{\Phi}(z)\mathbf{P}(z)\bar{\boldsymbol{\Phi}}(-z). \tag{27}$$

It is observed that $\mathbf{G}^{-1}(\mu) = \overline{\mathbf{G}(\mu)}$ and consequently that SIEWERT and BURNISTON's (1972) Theorem 2 is applicable here: there exists a canonical solution $\boldsymbol{\Phi}_1(z)$, of ordered normal form at infinity, to equation (22) such that

$$\overline{\boldsymbol{\Phi}_1(\bar{z})} = \boldsymbol{\Phi}_1(z). \tag{28}$$

Since $\mathbf{G}^{-1}(\mu) = \overline{\mathbf{G}(\mu)}$, it is clear that if $\boldsymbol{\Phi}(z)$ is a canonical solution of equation (22) then $\overline{\boldsymbol{\Phi}(\bar{z})}$ is a solution, and thus $\overline{\boldsymbol{\Phi}(\bar{z})} = \boldsymbol{\Phi}(z)\hat{\mathbf{P}}(z)$, with $\hat{\mathbf{P}}(z)$ being a polynomial matrix. We can now argue that there exists a canonical solution $\boldsymbol{\Phi}_1(z)$ such that the resulting $\hat{\mathbf{P}}(z) = \mathbf{I}$, which proves equation (28). It follows that $\boldsymbol{\Phi}_1(z)$ is real on the entire axis complementary to (0, 1] and

$$\boldsymbol{\Phi}_1(z)\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} \to \mathbf{A}_1, \quad \text{as} \quad |z| \to \infty, \tag{29}$$

where $\mathbf{A}_1$ is real, $A_1 = \det \mathbf{A}_1$ is non-zero and $\kappa_2 \geq \kappa_1$, $\kappa_1 + \kappa_2 = \kappa$. If we now let $\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}_1(z)$ and $\mathbf{W}(z) = \mathbf{C}\boldsymbol{\Lambda}(z)$, then equation (27) can be solved for $P_{11}(z)$, the polynomial in the 11-position of the resulting $\mathbf{P}(z)$:

$$P_{11}(z) = \frac{1}{\phi_1(z)\phi_1(-z)}[W_{11}(z)\phi_{122}(z)\phi_{122}(-z) + W_{22}(z)\phi_{112}(z)\phi_{112}(-z) - W_{12}(z)\phi_{122}(z)\phi_{112}(-z) - W_{12}(z)\phi_{112}(z)\phi_{122}(-z)], \tag{30}$$

where $\phi_{1\alpha\beta}(z)$ denotes the element in the $\alpha\beta$-position of $\boldsymbol{\Phi}_1(z)$ and $\phi_1(z) = \det \boldsymbol{\Phi}_1(z)$. From equation (30) it follows, after use is made of equation (29), that

$$z^{-2\kappa_1}P_{11}(z) \to \frac{(-1)^{\kappa_1}}{A_1^2}[W_{11}(\infty)A_{122}^2 + W_{22}(\infty)A_{112}^2 - 2W_{12}(\infty)A_{112}A_{122}], \quad \text{as} \quad |z| \to \infty. \tag{31}$$

We conclude from equation (31) that either the polynomial $P_{11}(z) \equiv 0$ or, if $P_{11}(z) \not\equiv 0$, that $\kappa_1 \geq 0$. In Appendix A we show (for the considered cases) that $P_{11}(z) \not\equiv 0$, and thus it follows that $\kappa_1 \geq 0$. Since the partial indices have, without loss of generality, been ordered such that $\kappa_2 \geq \kappa_1$, it is clear that both partial indices are non-negative.

For the case of $\kappa = 1$ the partial indices are $\kappa_1 = 0$ and $\kappa_2 = 1$; however, for the case $\kappa = 2$ we consider the two possibilities $\kappa_1 = 0$, $\kappa_2 = 2$ and $\kappa_1 = \kappa_2 = 1$. Note (SIEWERT and SHIEH, 1967) that det $\mathbf{C} < 0 \Rightarrow \kappa = 1$.

Equation (17) yields the fact that $z\mathbf{N}(z)$ must be bounded as $|z|$ tends to infinity:

$$\lim_{|z|\to\infty} z\mathbf{N}(z) \triangleq \mathbf{N}, \tag{32}$$

whereas for $\mathbf{\Phi}(z) = \mathbf{\Phi}_1(z)$ equation (23) yields, after use is made of equation (29),

$$\mathbf{A}_1^{-1}\tilde{\mathbf{N}} = \lim_{|z|\to\infty} \frac{1}{2\pi i}\begin{bmatrix} z^{-\kappa_1} & 0 \\ 0 & z^{-\kappa_2}\end{bmatrix}\left[-\int_0^1 \mathbf{K}(\nu)\,\mathrm{d}\nu + z\boldsymbol{\vartheta}(z)\right]. \tag{33}$$

Since $\mathbf{A}_1^{-1}\tilde{\mathbf{N}}$ is bounded, it follows that

$$\boldsymbol{\vartheta}(z) = \begin{bmatrix} a\,\delta_{1,\kappa_1} & b\,\delta_{1,\kappa_1} \\ c + a\,z\,\delta_{2,\kappa_2} & d + bz\,\delta_{2,\kappa_2}\end{bmatrix}, \tag{34}$$

where a, b, c and d are constants and $\delta_{i,j}$ is the Kronecker delta. A variation of SIEWERT and BURNISTON's (1972) Theorem 3 is available here: if $\mathbf{\Phi}(z)$ is a canonical solution of ordered normal form at infinity to the Riemann problem defined by equation (22), then so is

$$\mathbf{\Phi}(z)\begin{bmatrix} l & q\,\delta_{1,\kappa_1} \\ r + sz(1-\delta_{1,\kappa_1}) + tz^2\,\delta_{2,\kappa_2} & m\end{bmatrix}, \qquad lm - qr\,\delta_{1,\kappa_1} \neq 0, \tag{35}$$

where l, m, q, r, s and t are constants. We observe, in fact, that a canonical matrix of ordered normal form at infinity is unique to within a post-multiplication of the form given above. It can thus be concluded that $\tilde{\mathbf{N}}(z)$ as given by equation (23) with $\boldsymbol{\vartheta}(z)$ given by equation (34) is unique to within the two (four) degrees of freedom corresponding to the arbitrary constants c and d (a, b, c and d) for $\kappa = 1 (\kappa = 2)$.

The constraints given by equation (12c) and written, after use is made of equation (17), as

$$[\mathbf{I} + 2\pi i\nu_i\mathbf{N}(\nu_i)]\mathbf{M}(\nu_i) = \mathbf{0}, \qquad i = 1, 2, \ldots, \kappa, \tag{36}$$

can now be used to specify uniquely the constants in the $\boldsymbol{\vartheta}$-matrix. We find therefore a unique solution for $\mathbf{N}(z)$ and since $\mathbf{H}(\mu)$ follows from equation (19) the proof of the theorem is established.

3. CANONICAL SOLUTIONS AND THE H-MATRIX

Having proved the required existence and uniqueness theorem concerning the **H**-matrix for the two-group model, we now relate, in the manner of SIEWERT and BURNISTON (1972), the **H**-matrix to a certain canonical solution of the Riemann problem defined by equation (22). In addition to providing a derivation, alternative to that available from pursuit of CHANDRASEKHAR's (1950) invariance principles (at least for non-multiplying media), of several useful **H**-matrix identities and equations convenient for computational purposes, we note that the ensuing analysis reveals the basic character of the **H**-matrix with regard to real or complex values. The need to know for which cases and in what manner the **H**-matrix becomes complex is, of course, basic to any efficient method for computing $\mathbf{H}(\mu)$, $\mu \in [0, 1]$.

Consider first the case $\kappa = 1$ and a subsequent factorization of $\mathbf{\Lambda}(z)$:

$$\mathbf{C\Lambda}(z) = \mathbf{\Phi}_1(z)\mathbf{P}(z)\tilde{\mathbf{\Phi}}_1(-z), \tag{37}$$

where $\mathbf{\Phi}_1(z) = \overline{\mathbf{\Phi}_1(\bar{z})}$ is a canonical solution of ordered normal form at infinity,

$$\mathbf{\Phi}_1(z)\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} \to \mathbf{A}_1 \quad \text{as} \quad |z| \to \infty, \tag{38}$$

and

$$\mathbf{P}(z) = \begin{bmatrix} \alpha & \beta + \gamma z \\ \beta - \gamma z & \delta + \varepsilon z^2 \end{bmatrix}, \qquad \kappa_1 = 0, \qquad \kappa_2 = 1. \tag{39}$$

Here the constants α, β, γ, δ and ε are real, since $\mathbf{P}(z) = \overline{\mathbf{P}(\bar{z})}$, and

$$(\alpha\delta - \beta^2) + (\alpha\varepsilon + \gamma^2)z^2 = \frac{C\Lambda(\infty)}{A_1^{\,2}}(\nu_1^{\,2} - z^2). \tag{40}$$

If we now introduce

$$\mathbf{\Phi}_0(z) = \mathbf{\Phi}_1(z)\begin{bmatrix} |\alpha|^{1/2} & 0 \\ \dfrac{|\alpha|^{1/2}}{\alpha}(\beta - \gamma z) & \dfrac{|\alpha C|^{1/2}}{\alpha A_1}\nu_1[\Lambda(\infty)]^{1/2} \end{bmatrix}, \qquad \kappa_1 = 0, \qquad \kappa_2 = 1, \tag{41}$$

then equation (37) can be written as

$$\mathbf{C\Lambda}(z) = \frac{\alpha}{|\alpha|}\mathbf{\Phi}_0(z)\begin{bmatrix} 1 & 0 \\ 0 & C^{1/2}|C|^{-1/2} \end{bmatrix}\mathbf{D}(z)\mathbf{D}(-z)\begin{bmatrix} 1 & 0 \\ 0 & C^{1/2}|C|^{-1/2} \end{bmatrix}\tilde{\mathbf{\Phi}}_0(-z),$$
$$\kappa_1 = 0, \qquad \kappa_2 = 1, \tag{42}$$

where

$$\mathbf{D}(z) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\nu_1 - z}{\nu_1} \end{bmatrix}, \qquad \kappa_1 = 0, \qquad \kappa_2 = 1. \tag{43}$$

Note that $\mathbf{\Phi}_0(z)$ is also a canonical solution of ordered normal form at infinity and that $\mathbf{\Phi}_0(z) = \overline{\mathbf{\Phi}_0(\bar{z})}$, so that $\mathbf{\Phi}_0(z)$ is real on the entire real axis complementary to (0, 1].

We now let

$$\mathbf{H}(z) = \mathbf{C}\tilde{\mathbf{\Phi}}_0^{-1}(-z)\mathbf{D}^{-1}(-z)\tilde{\mathbf{\Phi}}_0(0)\mathbf{C}^{-1}, \qquad \kappa_1 = 0, \qquad \kappa_2 = 1, \tag{44}$$

so that equation (42) can be written as

$$\mathbf{\Lambda}(z) = \tilde{\mathbf{H}}^{-1}(-z)\mathbf{C}^{-1}\mathbf{H}^{-1}(z)\mathbf{C}. \tag{45}$$

Equation (44) is an extension to the complex plane of

$$\mathbf{\Theta}(\mu)\mathbf{H}(\mu) = \mathbf{\Theta}(\mu)\mathbf{C}\tilde{\mathbf{\Phi}}_0^{-1}(-\mu)\mathbf{D}^{-1}(-\mu)\tilde{\mathbf{\Phi}}_0(0)\mathbf{C}^{-1},$$
$$\mu \in [0, 1], \qquad \kappa_1 = 0, \qquad \kappa_2 = 1, \tag{46}$$

which is the unique solution to equations (12). We note that equation (46) can be obtained from equation (19), with $\mathbf{N}(z)$ as given by equation (23). Since $\mathbf{\Phi}_0(-\mu)$, $\mu \in [0, 1]$, is real for all cases, equation (46) for ν_1 real clearly yields a real $\mathbf{H}(\mu)$. For ν_1 imaginary, however, the matrix $\mathbf{D}(-\mu)$ is complex and thus the **H**-matrix resulting from equation (46) is also complex.

Further, equations (17), (42) and (44) can be used in the Cauchy representation

$$\mathbf{\Phi}_0(z) = \mathbf{\Phi}_0(\infty) + \frac{1}{2\pi i}\int_0^1 [\mathbf{\Phi}_0^{+}(\mu) - \mathbf{\Phi}_0^{-}(\mu)]\frac{d\mu}{\mu - z} \tag{47}$$

to yield

$$\mathbf{H}(z) = \mathbf{I} + z\mathbf{H}(z)\mathbf{C}\int_0^1 \tilde{\mathbf{H}}(\mu)\mathbf{\Theta}(\mu)\frac{d\mu}{\mu + z}, \tag{48}$$

or

$$\mathbf{H}(\mu) = \mathbf{I} + \mu\mathbf{H}(\mu)\mathbf{C}\int_0^1 \tilde{\mathbf{H}}(\mu')\mathbf{\Theta}(\mu')\frac{d\mu'}{\mu' + \mu}, \qquad \mu \in [0, 1], \tag{49}$$

the non-linear **H**-matrix equation useful for computational purposes (Siewert and Ishiguro, 1972) when used in conjunction with the constraint, equation (12c).

For the case $\kappa = 2$, a slight modification of the foregoing procedure is required since we wish to include both possibilities $\kappa_1 = 0$, $\kappa_2 = 2$ and $\kappa_1 = \kappa_2 = 1$. In general we observe from equation (37) that, since $\mathbf{C\Lambda}(\infty)$ is bounded and $\mathbf{\Lambda}(z) = \mathbf{\Lambda}(-z)$,

$$\mathbf{P}(z) = \begin{bmatrix} P_{2\kappa_1}(z) & P_\kappa(z) \\ P_\kappa(-z) & P_{2\kappa_2}(z) \end{bmatrix}, \tag{50}$$

where the subscripts explicitly denote the degrees of the polynomial entries. It follows for the case $\kappa_1 = 0$, $\kappa_2 = 2$, since $\mathbf{P}(z) = \overline{\mathbf{P}(\bar{z})}$ and $\tilde{\mathbf{P}}(z) = \mathbf{P}(-z)$, that $\mathbf{P}(z)$ is of the form

$$\mathbf{P}(z) = \begin{bmatrix} \alpha & (\beta_0 + \gamma_0 z)(\beta_1 + \gamma_1 z) \\ (\beta_0 - \gamma_0 z)(\beta_1 - \gamma_1 z) & \delta + \varepsilon z^2 + \zeta z^4 \end{bmatrix}, \qquad \kappa_1 = 0, \qquad \kappa_2 = 2. \tag{51}$$

We note that equation (37) can be evaluated at $z = 0$ to show that $\alpha > 0$, and thus we can use

$$\mathbf{\Phi}_0(z) = \mathbf{\Phi}_1(z)\begin{bmatrix} \sqrt{\alpha} & 0 \\ \dfrac{1}{\sqrt{\alpha}}(\beta_0 - \gamma_0 z)(\beta_1 - \gamma_1 z) & \dfrac{1}{\sqrt{\alpha}}\dfrac{1}{A_1}\nu_1\nu_2[C\Lambda(\infty)]^{1/2} \end{bmatrix}, \qquad \kappa_1 = 0, \qquad \kappa_2 = 2, \tag{52}$$

and equation (51) in equation (37) to obtain

$$\mathbf{C\Lambda}(z) = \mathbf{\Phi}_0(z)\mathbf{D}(z)\mathbf{D}(-z)\tilde{\mathbf{\Phi}}_0(-z), \qquad \kappa_1 = 0, \qquad \kappa_2 = 2, \tag{53}$$

where

$$\mathbf{D}(z) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{(\nu_1 - z)(\nu_2 - z)}{\nu_1\nu_2} \end{bmatrix}, \qquad \kappa_1 = 0, \qquad \kappa_2 = 2. \tag{54}$$

We therefore find that the definition

$$\mathbf{H}(z) = \mathbf{C}\tilde{\mathbf{\Phi}}_0^{-1}(-z)\mathbf{D}^{-1}(-z)\tilde{\mathbf{\Phi}}_0(0)\mathbf{C}^{-1}, \qquad \kappa_1 = 0, \qquad \kappa_2 = 2, \tag{55}$$

along with the definitions given by equations (52) and (54), renders equations (45), (48) and (49) valid also for the case $\kappa_1 = 0$, $\kappa_2 = 2$. For real ν_1 and ν_2, equation (55) clearly yields a real H-matrix on the real axis complementary to (0, 1], since $\mathbf{\Phi}_0(z) = \overline{\mathbf{\Phi}_0(\bar{z})}$.

For the case $\kappa_1 = \kappa_2 = 1$, we note that equation (35) allows us to specify explicitly, at any point in the cut plane, the value of a canonical matrix with ordered normal form at infinity. We thus let

$$\mathbf{\Phi}_0(z) = \mathbf{\Phi}_1(z)\mathbf{\Phi}_1^{-1}(0)\mathbf{Q}, \qquad \kappa_1 = \kappa_2 = 1, \tag{56}$$

where

$$\mathbf{Q} = \frac{1}{\sqrt{(c_{22})}}\begin{bmatrix} \sqrt{C} & c_{12} \\ 0 & c_{22} \end{bmatrix}, \qquad \mathbf{C} = \mathbf{Q}\tilde{\mathbf{Q}}, \tag{57}$$

and therefore write equation (37) as

$$\mathbf{C\Lambda}(z) = \mathbf{\Phi}_0(z)\mathbf{P}(z)\tilde{\mathbf{\Phi}}_0(-z). \tag{58}$$

Equation (50), along with the facts that $\mathbf{\Phi}_0(z) = \overline{\mathbf{\Phi}_0(\bar{z})}$, $\mathbf{P}(z) = \overline{\mathbf{P}(\bar{z})}$ and $\mathbf{P}(z) = \tilde{\mathbf{P}}(-z)$, can now be used to deduce the form of $\mathbf{P}(z)$:

$$\mathbf{P}(z) = \mathbf{I} + \gamma\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}z - \mathbf{\Gamma}z^2, \qquad \kappa_1 = \kappa_2 = 1, \tag{59}$$

where γ is real and $\mathbf{\Gamma} = \tilde{\mathbf{\Gamma}}$ is real. Equation (58) yields

$$\mathbf{\Gamma} = \mathbf{A}_0^{-1}\mathbf{C\Lambda}(\infty)\tilde{\mathbf{A}}_0^{-1} \tag{60}$$

where

$$\mathbf{A}_0 = \lim_{|z|\to\infty} z\mathbf{\Phi}_0(z). \tag{61}$$

Note that $\mathbf{A}_0$ is real.

It can be shown that equation (59) can be factored in the form

$$\mathbf{P}(z) = \mathbf{\Delta}(z)\tilde{\mathbf{\Delta}}(-z), \qquad \kappa_1 = \kappa_2 = 1, \tag{62}$$

where

$$\mathbf{\Delta}(z) = \mathbf{I} - z\boldsymbol{\delta}, \qquad \kappa_1 = \kappa_2 = 1. \tag{63}$$

We have also concluded that there exists a unique constant matrix $\boldsymbol{\delta}$, the solution to

$$\boldsymbol{\delta}\tilde{\boldsymbol{\delta}} = \mathbf{\Gamma} \quad \text{and} \quad \tilde{\boldsymbol{\delta}} - \boldsymbol{\delta} = \gamma\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{64}$$

such that

$$\det \mathbf{\Delta}(\nu_\alpha) = 0, \qquad \alpha = 1 \text{ and } 2, \tag{65}$$

where $\pm\nu_1$ and $\pm\nu_2$ are the zeros of $\det \mathbf{P}(z)$. We maintain here the convention that ν_α denotes an eigenvalue with a positive real (imaginary) part. It follows that

$$\mathbf{C\Lambda}(z) = \mathbf{\Phi}_0(z)\,\mathbf{\Delta}(z)\,\tilde{\mathbf{\Delta}}(-z)\tilde{\mathbf{\Phi}}_0(-z), \qquad \kappa_1 = \kappa_2 = 1. \tag{66}$$

For this case we can show that the unique **H**-matrix is given by

$$\mathbf{H}(z) = \mathbf{C}\bar{\mathbf{\Phi}}_0^{-1}(-z)\tilde{\mathbf{\Delta}}^{-1}(-z)\bar{\mathbf{\Phi}}_0(0)\mathbf{C}^{-1}, \qquad \kappa_1 = \kappa_2 = 1, \tag{67a}$$

or

$$\mathbf{H}(\mu) = \mathbf{C}\bar{\mathbf{\Phi}}_0^{-1}(-\mu)\tilde{\mathbf{\Delta}}^{-1}(-\mu)\mathbf{\Phi}_0(0)\mathbf{C}^{-1}, \qquad \mu \in [0, 1], \qquad \kappa_1 = \kappa_2 = 1. \tag{67b}$$

Since $\bar{\mathbf{\Phi}}_0^{-1}(-\mu)$, $\mu \in [0, 1]$, is real, the real or complex nature of $\mathbf{H}(\mu)$ is determined by equations (64), (65) and (67b). When ν_1 and ν_2 are both real, we have proved that the $\mathbf{\delta}$-matrix is real and subsequently that $\mathbf{H}(\mu)$, $\mu \in [0, 1]$, is real.

Finally we find that equations (63) and (67a) can be used to show that equations (45), (48) and (49) are valid also for this case, $\kappa_1 = \kappa_2 = 1$. Equations (45), (48) and (49) *are therefore independent of the partial indices.*

In Section 2 we established proof of the existence of a unique solution to equations (12) and subsequently developed equation (49) specifically to be used with equation (12c) for computational purposes. It thus follows that we must show that equations (49) and (12c) possess a *unique* solution. Since equations (12) possess a unique solution, we need simply show that any solution of equations (49) and (12c) is also a solution of equations (12).

We first write equation (49) as

$$\mathbf{H}(\mu)\left[\mathbf{I} - \mu\mathbf{C}\int_0^1 \tilde{\mathbf{H}}(\mu')\mathbf{\Theta}(\mu')\frac{d\mu'}{\mu' + \mu}\right] = \mathbf{I}, \qquad \mu \in [0, 1], \tag{68a}$$

or, alternatively,

$$\left[\mathbf{I} - \mu\mathbf{C}\int_0^1 \tilde{\mathbf{H}}(\mu')\mathbf{\Theta}(\mu')\frac{d\mu'}{\mu' + \mu}\right]\mathbf{H}(\mu) = \mathbf{I}, \qquad \mu \in [0, 1]. \tag{68b}$$

If we now multiply the *transpose* of equation (68b) from the right by

$$\mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\mu')\mathbf{\Psi}(\mu')\frac{d\mu'}{\mu' - \mu},$$

do some partial-fraction analysis and invoke equations (15) and (68), then we find

$$\tilde{\mathbf{H}}(\mu)\mathbf{\lambda}(\mu) = \mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\mu')\mathbf{\Psi}(\mu')\frac{d\mu'}{\mu' - \mu}, \qquad \mu \in (0, 1), \tag{69}$$

which proves that all solutions of equation (49) are also solutions of equations (12a) and (12b) and thus that the solution of equations (49) *and* (12c) is unique.

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APPENDIX A

Partial indices

As discussed in Section 2, our proof that the partial indices of the canonical solution $\mathbf{\Phi}(z)$ are non-negative requires proof that the polynomial $P_{11}(z)$ given by equation (30) is not identically zero. A similar proof is required in the discussion of partial indices given by Kuščer (1967).

The case det $\mathbf{C} > 0$ can be resolved immediately by setting $z = 0$ in equation (30) and recalling that since

$$\overline{\mathbf{\Phi}_1(\bar{z})} = \mathbf{\Phi}_1(z) \tag{A.1}$$

$\mathbf{\Phi}_1(0)$ is real:

$$P_{11}(0) = \frac{1}{\phi_1^{\,2}(0)}\,[c_{11}\phi_{122}^2(0) + c_{22}\phi_{112}^2(0) - 2c_{12}\phi_{122}(0)\phi_{112}(0)], \tag{A.2}$$

or alternatively

$$P_{11}(0) = \frac{1}{c_{22}\phi_1^{\,2}(0)}\,([\sqrt{C}\phi_{122}(0)]^2 + [c_{12}\phi_{122}(0) - c_{22}\phi_{112}(0)]^2). \tag{A.3}$$

Thus for det $\mathbf{C} > 0$, $P_{11}(0) > 0$.

We consider now the case det $\mathbf{C} < 0$. For $\mathbf{\Phi}(z) = \mathbf{\Phi}_1(z)$, equation (27) can be written as

$$\mathbf{C}\mathbf{\Lambda}(z) = \mathbf{\Phi}_1(z)\,\mathbf{P}(z)\,\tilde{\mathbf{\Phi}}_1(-z), \tag{A.4}$$

where, since

$$\mathbf{\Lambda}(z) = \mathbf{\Lambda}(-z), \qquad \mathbf{C}\mathbf{\Lambda}(z) = \tilde{\mathbf{\Lambda}}(z)\mathbf{C} \quad \text{and} \quad \mathbf{\Lambda}(z) = \overline{\mathbf{\Lambda}(\bar{z})}, \tag{A.5}$$

we conclude that

$$\mathbf{P}(z) = \tilde{\mathbf{P}}(-z) \text{ and } \overline{\mathbf{P}(\bar{z})} = \mathbf{P}(z). \tag{A.6}$$

Further, since

$$\det \mathbf{P}(z) \propto (\nu_1^{\,2} - z^2), \tag{A.7}$$

the proposition that $P_{11}(z) \equiv 0$ implies that $\mathbf{P}(z)$ is the form

$$\mathbf{P}(z) = \begin{bmatrix} 0 & a + bz \\ a - bz & P_{22}(z) \end{bmatrix} \tag{A.8}$$

where $(a/b)^2 = \nu_1^{\,2}$. Since the constants a and b must, in view of the fact that $\overline{\mathbf{P}(\bar{z})} = \mathbf{P}(z)$, be real, the conclusion that $(a/b)^2 = \nu_1^{\,2}$ is obviously a contradiction if ν_1 is imaginary. It follows that $P_{11}(z)$ cannot be identically zero for det $\mathbf{C} < 0$ *and* ν_1 imaginary $[\Lambda(\infty) < 0]$.

Seeking a contradiction to the form of $\mathbf{P}(z)$ given by equation (A.8) has not led to conclusive results for the remaining case det $\mathbf{C} < 0$ *and* $\Lambda(\infty) > 0$; however, for at least a subset of the elusive cases, the proposition that $P_{11}(z) \equiv 0$ can be shown to lead to a contradiction. If we denote the second column of $\mathbf{\Phi}_1(z)$ by $\mathbf{\Phi}_1^{\,2}(z)$, then by Cauchy's theorem we can write

$$\mathbf{\Phi}_1^{\,2}(z) = \frac{1}{2\pi i}\int_0^1 [\mathbf{\Phi}_1^{\,2+}(\mu) - \mathbf{\Phi}_1^{\,2-}(\mu)]\,\frac{d\mu}{\mu - z}\,. \tag{A.9}$$

If we now make use of equations (21), (22), (A.8) and (A.4) then equation (A.9) for $z \in [-1, 0]$ yields the integral equation

$$\mathbf{F}(\mu) = C\int_0^1 \nu\mathbf{\Theta}(\nu)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} f(\nu)\mathbf{F}(\nu)\,\frac{d\nu}{\nu + \mu}\,, \quad \mu \in [0, 1], \tag{A.10}$$

for

$$\mathbf{F}(\mu) = \mathbf{C}^{-1}\mathbf{\Phi}_1{}^2(-\mu). \tag{A.11}$$

Here $\mathbf{\Theta}(\mu)$ is given by equation (11) and

$$f(\nu) = \frac{1}{(\pm\nu_1 + \nu)X(-\nu)K}, \tag{A.12}$$

where

$$X(-\nu) = \frac{1}{1+\nu}\exp\frac{1}{\pi}\int_0^1 \arg\Lambda^+(\mu)\frac{\mathrm{d}\mu}{\mu+\nu} \tag{A.13}$$

and $K^2 = -C\Lambda(\infty)$. Equation (A.10) can be uncoupled to yield two scalar equations for the components $F_1(\mu)$ and $F_2(\mu)$ of $\mathbf{F}(\mu)$:

$$F_1(\mu) = \int_0^1 F_1(\mu')K_1(\mu' \to \mu)\,\mathrm{d}\mu', \qquad \mu \in [0, 1], \tag{A.14a}$$

and

$$F_2(\mu) = \int_0^{1/\sigma} F_2(\mu')K_2(\mu' \to \mu)\,\mathrm{d}\mu', \qquad \mu \in [0, 1/\sigma], \tag{A.14b}$$

where

$$K_1(\mu' \to \mu) = -C^2\mu' f(\mu')\int_0^{1/\sigma}\frac{\nu f(\nu)}{(\nu+\mu')(\nu+\mu)}\,\mathrm{d}\nu \tag{A.15a}$$

and

$$K_2(\mu' \to \mu) = -C^2\mu' f(\mu')\int_0^1\frac{\nu f(\nu)}{(\nu+\mu')(\nu+\mu)}\,\mathrm{d}\nu. \tag{A.15b}$$

It is clear that if we could show that either of equations (A.14) yielded only the trivial solution then we would have the contradiction, through equation (A.10), required to prove $P_{11}(z) \not\equiv 0$. It is also apparent that the homogeneous equation (A.14b) would yield only trivial solutions if the L_2 norm $\|K_2\|$ of $K_2(\mu' \to \mu)$ were less than unity:

$$\|K_2\|^2 = \int_0^{1/\sigma}\int_0^{1/\sigma} K_2(\mu' \to \mu)K_2(\mu' \to \mu)\,\mathrm{d}\mu'\,\mathrm{d}\mu. \tag{A.16}$$

We have not been able to resolve the ($\pm$) sign appearing explicitly in equation (A.12); thus to seek an upper bound to $\|K_2\|$ we consider the worst possibility—the minus sign in equation (A.12). We find

$$\|K_2\|^2 < \frac{C^2}{\Lambda^2(\infty)}\frac{1}{\sigma}[J_1(\sigma,\nu_1) + J_2(\sigma,\nu_1) + J_3(\sigma,\nu_1)], \tag{A.17}$$

where

$$J_1(\sigma,\nu_1) = (\nu_1+1)^2\left(\ln\frac{\nu_1}{\nu_1-1}\right)^2\left\{\frac{1}{2\sigma}\left[\frac{(\sigma+1)^2}{\sigma^2\nu_1{}^2-1}+3\right] + \ln\frac{\sigma^2\nu_1{}^2-1}{\sigma^2\nu_1{}^2} - \frac{1}{2\nu_1}[1+3\nu_1{}^2]\tanh^{-1}\frac{1}{\sigma\nu_1}\right\}, \tag{A.18}$$

$$J_2(\sigma,\nu_1) = 2\,\frac{\nu_1+1}{\sigma}\,\ln\frac{\nu_1}{\nu_1-1}\ln(1+\sigma)\left[\frac{3\nu_1{}^2-1}{2\nu_1}\tanh^{-1}\frac{1}{\sigma\nu_1} + \tfrac{1}{2}\ln\frac{\sigma^2\nu_1{}^2}{\sigma^2\nu_1{}^2-1} + \frac{(\sigma^2-1)(1+\sigma)}{2\sigma(\sigma^2\nu_1{}^2-1)} - \frac{3}{2\sigma} - \frac{1}{2\nu_1{}^2}\right] \tag{A.19}$$

and

$$J_3(\sigma,\nu_1) = \frac{1}{\sigma^2}[\ln(1+\sigma)]^2\left[\frac{(1-\nu_1{}^2)(1+3\nu_1{}^2)}{2\nu_1{}^3}\tanh^{-1}\frac{1}{\sigma\nu_1} + \frac{1}{\sigma} + \frac{\sigma(1-\nu_1{}^2)^2}{2\nu_1{}^2(\sigma^2\nu_1{}^2-1)}\right]. \tag{A.20}$$

Though equation (A.17) is unbounded in the limit as both $\nu_1 \to 1$ and $\sigma \to 1$, it is apparent that equation (A.17) can in fact be used to show that $\|K_2\| < 1$ for some values of σ and **C** for which det **C** < 0 *and* $\Lambda(\infty) > 0$. Unfortunately this procedure does not ensure that $P_{11}(z) \not\equiv 0$ for all the cases for which det **C** < 0 *and* $\Lambda(\infty) > 0$, but equation (A.17) can be used to test particular cases of this class and thus to prove for particular cases the desired result: $P_{11}(z) \not\equiv 0$. We have evaluated equations (A.18), (A.19) and (A.20) for selected data sets and have established numerically for these cases that $\|K_2\| < 1$.

APPENDIX B

The half-range expansion theorem

Since the proof of the following theorem is essentially equivalent to the proof of the existence of a unique solution to equations (12), we shall only outline the salient points of the proof:

Theorem. *For* det **C** ≥ 0 *and for all* det **C** < 0 *cases for which* $\Lambda(\infty) \leq 0$, *the eigenvectors* $\mathbf{F}(\nu_i, \mu)$, $i = 1, 2 \ldots \kappa$, $\kappa = 1$ *or* 2, $\mathbf{F}_1^{(1)}(\nu, \mu)$, $\mathbf{F}_2^{1)}(\nu, \mu)$ *and* $\mathbf{F}^{(2)}(\nu, \mu)$, $\nu \in (0, 1)$, *form a complete basis for the expansion of continuous two-vectors* $\mathbf{F}(\mu)$ *in the sense that*

$$\mathbf{F}(\mu) = \sum_{i=1}^{\kappa} A(\nu_i)\mathbf{F}(\nu_i, \mu) + \sum_{i=1}^{2} \int_0^{1/\sigma} A_i^{(1)}(\nu)\mathbf{F}_i^{(1)}(\nu, \mu)\, d\nu + \int_{1/\sigma}^1 A^{(2)}(\nu)\mathbf{F}^{(2)}(\nu, \mu)\, d\nu, \quad \mu \in (0, 1). \tag{B.1}$$

If we let

$$\mathbf{L}(z) = \frac{1}{2\pi i}\int_0^1 \nu\mathbf{A}(\nu)\frac{d\nu}{\nu - z}, \tag{B.2}$$

where

$$\mathbf{A}(\nu) = [A_1^{(1)}(\nu)\mathbf{M}_1^{(1)}(\nu) + A_2^{(1)}(\nu)\mathbf{M}_2^{(1)}(\nu)]\theta(\nu) + A^{(2)}(\nu)\mathbf{M}^{(2)}(\nu)[1 - \theta(\nu)], \tag{B.3}$$

with, in general,

$$\mathbf{M}(\xi) = \int_{-1}^1 \mathbf{F}(\xi, \mu)\, d\mu, \tag{B.4}$$

then equation (B.1) can be expressed in the form

$$\mu\Sigma\mathbf{F}''(\mu) = \mathbf{\Lambda}^+(\mu)\mathbf{L}^+(\mu) - \mathbf{\Lambda}^-(\mu)\mathbf{L}^-(\mu), \qquad \mu \in (0, 1), \tag{B.5}$$

where

$$\mathbf{F}''(\mu) = \mathbf{\Theta}(\mu)\begin{vmatrix} F_1'(\sigma\mu) \\ F_2'(\mu) \end{vmatrix} \tag{B.6}$$

and

$$\mathbf{F}'(\mu) = \mathbf{F}(\mu) - \sum_{i=1}^{\kappa} A(\nu_i)\mathbf{F}(\nu_i, \mu). \tag{B.7}$$

We can now write the solution to equation (B.5) as

$$\mathbf{L}(z) = \tilde{\mathbf{\Phi}}^{-1}(z)\left[\frac{1}{2\pi i}\int_0^1 \mu\tilde{\mathbf{\Phi}}^+(\mu)[\mathbf{\Lambda}^+(\mu)]^{-1}\Sigma\mathbf{F}''(\mu)\frac{d\mu}{\mu - z} + \mathbf{P}_*(z)\right], \tag{B.8}$$

where $\mathbf{\Phi}(z)$ is a canonical solution of equation (22) and $\mathbf{P}_*(z)$ is a matrix of polynomials. Now since, as shown in Appendix A, the partial indices associated with $\mathbf{\Phi}(z)$ are non-negative, and since from equation (B.2) $z\mathbf{L}(z)$ must be bounded as $|z| \to \infty$, we conclude that $\mathbf{P}_*(z) \equiv 0$ and, further, that $\mathbf{F}''(\mu)$ must be restricted such that

$$\lim_{z\to\infty}\begin{vmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{vmatrix}\int_0^1 \mu\tilde{\mathbf{\Phi}}^+(\mu)[\mathbf{\Lambda}^+(\mu)]^{-1}\Sigma\mathbf{F}''(\mu)[1 + \mu/z]\, d\mu < \infty. \tag{B.9}$$

It can be shown that by choosing correctly the discrete expansion coefficients $A(\nu_i)$ in equation (B.6) that equation (B.9) can be satisfied for all continuous $\mathbf{F}(\mu)$, for any of the possible pairs of partial indices. Of course, for the case $\kappa = 2$, one must show that the two simultaneous equations for $A(\nu_1)$ and $A(\nu_2)$ are solvable, which can, in fact, be done.