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Anisotropic Scattering Coefficients in the Constant Cross-Section Transport Equation

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The coefficients that enter the anisotropic scattering formulation of the constant cross-section transport equation are investigated. Exact expressions to order ten are presented as well as a useful, nested series for numerical calculations to any order.

The scattering probability for anisotropic elastic collisions of neutrons with stationary nuclei in the speed-independent transport equation is given by Davison[1] as

$$f(\mathbf{\Omega}' \to \mathbf{\Omega}) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} b_\ell P_\ell(\mathbf{\Omega} \cdot \mathbf{\Omega}') \tag{1}$$

where the coefficients are

$$b_\ell = (2A)^{-1} \int_{-1}^{1} \frac{[z + (A^2 - 1 + z^2)^{1/2}]^2}{(A^2 - 1 + z^2)^{1/2}} P_\ell(z)\, dz, \tag{2}$$

with $A \geq 1$.

In this paper, methods for computing the b_ℓ exactly or in a series of powers of $1/A$ are developed for all ℓ to facilitate solution of the transport equation. The integral in Eq. (2) is rearranged to the form

$$b_\ell = A^{-1}\int_{-1}^{1} zP_\ell(z)\, dz + (2A)^{-1}\int_{-1}^{1} \frac{A^2 - 1 + 2z^2}{(A^2 - 1 + z^2)^{1/2}} P_\ell(z)\, dz\,. \tag{3}$$

Noting the odd and even character of the integrands

$$b_{2n+1} = A^{-1}\int_{-1}^{1} zP_{2n+1}(z)\, dz \tag{4a}$$

and

$$b_{2n} = (2A)^{-1}\int_{-1}^{1} \frac{A^2 - 1 + 2z^2}{(A^2 - 1 + z^2)^{1/2}} P_{2n}(z)\, dz\,. \tag{4b}$$

It is seen at once that

$$b_{2n+1} = \frac{2}{3A}\, \delta_{n,0}\,, \tag{5}$$

i.e., all odd coefficients for $\ell > 1$ are identically zero, with $b_1 = 2/3A$.

The Legendre polynomials of even degree may be represented generally by

$$P_{2n}(z) = \sum_{k=0}^{n} C_{nk} z^{2k}\,, \tag{6}$$

where[2,3]

$$C_{nk} = \frac{(-1)^{n-k}}{2^{2n}} \binom{2n}{n-k} \binom{2n+2k}{2k}\,. \tag{7}$$

Inserting Eq. (6) in Eq. (4b) and integrating by parts yields

$$b_{2n} = 1 - \sum_{k=1}^{n} 2k\, C_{nk} I_k\,, \tag{8}$$

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[1] B. DAVISON, *Neutron Transport Theory*, p. 232, Oxford University Press, London (1957).

[2] T. M. MacROBERT, *Spherical Haromonics*, 2nd Ed., p. 86, Dover Publications, New York.

[3] M. ABRAMOWITZ and I. A. STEGUN, Editors, *Handbook of Mathematical Functions*, p. 796, AMS55, National Bureau of Standards, Washington (1964).

where

$$I_k = A^{-1}\int_0^1 z^{2k} \ (A^2 - 1 + z^2)^{1/2} \ dz. \qquad (9)$$

The integrals are tabulated[4], and obey the recursion relation

$$I_k = \frac{A^2 - (2k-1)(A^2-1)I_{k-1}}{2k+2} \qquad (10a)$$

with

$$I_0 = \frac{1}{2} - \frac{(A^2-1)\ln\alpha}{8A}, \qquad (10b)$$

where $\alpha = (A-1/A+1)^2$. The exact values of the coefficients expressed by Eq. (4b) become

$$b_{2n} = (M_{2n})^{-1}\left[\sum_{k=0}^{n}(-1)^k \ C_k^{2n} \ A^{2k} + (-1)^n \frac{\ln\alpha}{4A}\sum_{k=2}^{n+1} L_k^{2n} \ (A^2-1)^k\right]. \qquad (11)$$

Numerical values for the C_k^{2n}, M_{2n}, and L_k^{2n} are given in Tables I and II. To illustrate the use of Eq. (11) and the Tables, the first three even coefficients are given explicitly:

$$b_0 = 1; \qquad (12)$$

$$b_2 = \frac{1}{8}\left\{5 - 3A^2 - \frac{\ln\alpha}{4A}\left[3(A^2-1)^2\right]\right\}; \qquad (13)$$

and

$$b_4 = \frac{1}{96}\left\{81 - 190A^2 + 105A^4 + \frac{\ln\alpha}{4A} \times \left[90(A^2-1)^2 + 105(A^2-1)^3\right]\right\}. \qquad (14)$$

A convenient series representation of the b_{2n} is obtained by expanding the integrand of I_k in Eq. (9) in powers of $1/A$. A set of integrals in terms of the beta function appears,

$$\int_0^1 z^{2m}(1-z^2)^k \ dz = \frac{1}{2}B\left(\frac{2m+1}{2}, \ k+1\right) = \frac{k! \ 2^{k-1}}{(2k+2m+1)(2k+2m-1)\ldots(2m+1)}. \qquad (15)$$

After much manipulation involving the Legendre coefficients in Eq. (7), the desired set of coefficients for $2n \geq 2$ is found from Eq. (8) to be

[4]W. GRÖBNER and N. HOFREITER, *Integraltafel* No. 234, 6a, Springer-Verlag, Wien (1961).

TABLE I

The Coefficients C_k^{2n} and M_{2n}

	C_k^{2n}						
$2n$ \ k	0	1	2	3	4	5	M_{2n}
0	1						1
2	5	3					8
4	81	190	105				96
6	919	5103	7665	3465			1024
8	3781	38 232	109 494	120 120	45 045		4096
10	368 961	5 907 275	27 353 898	52 954 902	45 690 645	14 549 535	393 216

TABLE II

The Coefficient L_k^{2n}

	L_k^{2n}				
$2n$ \ k	2	3	4	5	6
0					
2	3				
4	90	105			
6	1680	5040	3465		
8	10 080	55 440	90 090	45 045	
10	1 330 560	11 531 520	32 432 400	36 756 720	14 549 535

$$b_{2n} = \sum_{k=n}^{\infty} \frac{s_{k,2n}}{A^{2k}} \quad , \tag{16}$$

where

$$s_{n,2n} = (-1)^n \frac{(-1)1\cdot 3\cdots(2n-5)(2n-3)}{(2n+3)(2n+5)\cdots\cdot(4n+1)} , \tag{17a}$$

$$\frac{s_{k+1,2n}}{s_{k,2n}} = \frac{(2k)(2k-1)}{(2k-2n+2)(2k+2n+3)} \tag{17b}$$

and

$$\frac{s_{n+1,2n+2}}{s_{n,2n}} = \frac{-(2n-1)(2n+3)}{(4n+5)(4n+3)} \quad . \tag{17c}$$

The most useful form for computing the b_{2n} is the nested series, the first few of which are shown below to indicate the pattern:

$$b_2 = \frac{1}{5A^2} \quad \left(1 + \frac{1\cdot 2}{2\cdot 7A^2}\right. \quad \left(1 + \frac{3\cdot 4}{4\cdot 9A^2}\right. \quad \left(1 + \frac{5\cdot 6}{6\cdot 11A^2}\right. \quad (1+\cdots \tag{18a}$$

$$b_4 = \frac{-1\cdot 1}{7\cdot 9A^4} \quad \left(1 + \frac{3\cdot 4}{2\cdot 11A^2}\right. \quad \left(1 + \frac{5\cdot 6}{4\cdot 13A^2}\right. \quad \left(1 + \frac{7\cdot 8}{6\cdot 15A^2}\right. \quad (1+\cdots \tag{18b}$$

$$b_6 = \frac{1\cdot 3}{9\cdot 11\cdot 13A^6} \quad \left(1 + \frac{5\cdot 6}{2\cdot 15A^2}\right. \quad \left(1 + \frac{7\cdot 8}{4\cdot 17A^2}\right. \quad \left(1 + \frac{9\cdot 10}{6\cdot 19A^2}\right. \quad (1+\cdots \tag{18c}$$

$$b_8 = \frac{-1\cdot 3\cdot 5}{11\cdot 13\cdot 15\cdot 17A^8} \quad \left(1 + \frac{7\cdot 8}{2\cdot 19A^2}\right. \quad \left(1 + \frac{9\cdot 10}{4\cdot 21A^2}\right. \quad \left(1 + \frac{11\cdot 12}{6\cdot 23A^2}\right. \quad (1+\cdots \tag{18d}$$

and

$$b_{10} = \frac{1\cdot 3\cdot 5\cdot 7}{13\cdot 15\cdot 17\cdot 19\cdot 21A^{10}} \quad \left(1 + \frac{9\cdot 10}{2\cdot 23A^2}\right. \quad \left(1 + \frac{11\cdot 12}{4\cdot 25A^2}\right. \quad \left(1 + \frac{13\cdot 14}{6\cdot 27A^2}\right. \quad (1+\cdots \tag{18e}$$

The computation of approximate values for b_{2n} with partial sums of Eq. (16) or truncations of the nested series as shown in Eqs. (18) is facilitated by the following error estimate: Consider the form

$$b_{2n} = \frac{s_{n,2n}}{A^{2n}} \left[1 + r_1(1 + r_2(1 + r_3(1 + \ldots\right] . \tag{19}$$

If one truncates by setting $r_{m+1} = 0$, the error or remainder R_m *within the square brackets*, can be estimated as follows: If $m \geq n/(A^2 - 1)$,

$$R_m < (r_1 r_2 \ldots r_m)\left(\frac{n}{A^2 - 1}\right) . \tag{20}$$

The series form is preferable to the exact expressions of Eq. (11) in computations for most elements other than hydrogen, for which the expansion of Eq. (16) is invalid. For the sake of completeness, the values for the first few coefficients for mass ratio 1 are shown in Table III.

TABLE III

The b_ℓ values for $A = 1$

ℓ	0	1	2	4	6	8	10
b_ℓ	1	2/3	1/4	-1/24	1/64	-1/128	7/1536