Steady-State Solutions in the Two-Group Theory of Neutron Diffusion

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Functional analysis arguments are used to prove the existence of a unique solution to the integral form of the two-group neutron-transport equation for subcritical half-spaces. The analytic properties of the solutions are discussed and used to prove that the partial indices of canonical solutions of the matrix Riemann problem, basic to **H**-matrix or half-range completeness considerations, are nonnegative.

I. INTRODUCTION

There has been considerable interest[1-7] in recent years in the multigroup version of the neutron-transport equation, basically because a great deal of the fine structure of such energy-dependent processes as scattering, absorption, and fission can be maintained in this model without actually requiring solutions to the more general energy-dependent form of the transport equation. In fact, the multigroup model has proved adequate for so many reactor calculations that multigroup diffusion theory is perhaps the most widely used method in reactor design analysis. Since efficient multigroup diffusion codes often make use of transport theory, for example, to define improved boundary conditions, we develop here the fundamental analysis required to place the two-group model on a basis equally as firm as that provided by Case[8] for the one-speed theory.

It seems that even the very basic subcriticality conditions for infinite media have not been resolved definitively for the multigroup model, and hence in Sec. II, we first seek the general conditions required to ensure the existence of a unique solution to the half-space albedo problem, based on the two-group model. These same conditions can, naturally, be shown to be the infinite-media subcriticality conditions.

Our principal goal here is to point out that half-space problems in two-group transport theory can be reduced to a convenient computational form and to provide the appropriate existence and uniqueness theorems required to ensure that any computational results can be interpreted and used with confidence. We shall rely rather heavily on a previous paper,[7] hereafter referred to as SBK, in which the required analysis was given for all cases but one. In addition to resolving the one elusive case not included in SBK, we are confident that the functional analysis arguments developed in Sec. II, and used in Sec. III to establish definitively, in the manner of Goh'berg and Krein,[9] the very important proof that the partial indices of a canonical solution to a basic matrix Riemann problem are nonnegative, will prove very useful in the analysis of the more general models.

We consider the homogeneous, steady-state neutron-transport equation written in a convenient form as

$$\mu \frac{\partial}{\partial x} \Psi(x, \mu) + \Sigma \Psi(x, \mu) = \mathbf{C} \int_{-1}^{1} \Psi(x, \mu') d\mu', \tag{1}$$

where the two elements of $\Psi(x, \mu)$ are the angular fluxes in each of the two energy groups, and $\mathbf{C}$, with nonnegative elements, is the group-transfer matrix. By choosing to measure distances in terms of the optical variable x, defined in terms of σ_2 (the smaller of the two total cross sections σ_1 and σ_2), we can write

$$\mathbf{\Sigma} = \begin{vmatrix} \sigma & 0 \\ 0 & 1 \end{vmatrix}, \qquad \sigma = \frac{\sigma_1}{\sigma_2} > 1. \tag{2}$$

Since the solutions to typical half-space problems based on Eq. (1) can be expressed in terms of the solution to the albedo problem, we seek a bounded (at infinity) *matrix* solution of Eq. (1) subject to the boundary condition

$$\Psi(\mu_1, \mu_2; 0, \mu) = \begin{vmatrix} \delta(\mu - \mu_1) & 0 \\ 0 & \delta(\mu - \mu_2) \end{vmatrix}, \qquad \mu_1, \mu_2, \mu \in (0, 1). \tag{3}$$

We can now enter

$$\Psi(\mu_1, \mu_2; x, \mu) = \hat{\Psi}(\mu_1, \mu_2; x, \mu) + \begin{vmatrix} \delta(\mu - \mu_1) e^{-\sigma x/\mu} & 0 \\ 0 & \delta(\mu - \mu_2) e^{-x/\mu} \end{vmatrix} \tag{4}$$

into Eq. (1) to obtain

$$\mu \frac{\partial}{\partial x} \hat{\Psi}(\mu_1, \mu_2; x, \mu) + \Sigma \hat{\Psi}(\mu_1, \mu_2; x, \mu) = \mathbf{CF}(\mu_1, \mu_2; x), \tag{5}$$

where

$$\mathbf{F}(\mu_1, \mu_2; x) = \int_{-1}^{1} \hat{\Psi}(\mu_1, \mu_2; x, \mu) d\mu + \begin{vmatrix} e^{-\sigma x/\mu_1} & 0 \\ 0 & e^{-x/\mu_2} \end{vmatrix}; \tag{6}$$

from Eq. (3) we note that

$$\hat{\Psi}(\mu_1, \mu_2; 0, \mu) = \mathbf{0}, \qquad \mu \in (0, 1). \tag{7}$$

Equation (5) can be solved at once to yield

$$\hat{\Psi}(\mu_1, \mu_2; x, \mu) = \frac{1}{\mu} \int_0^x \begin{vmatrix} e^{-\sigma(x-x')/\mu} & 0 \\ 0 & e^{-(x-x')/\mu} \end{vmatrix} \times \mathbf{CF}(\mu_1, \mu_2; x') dx', \quad \mu > 0, \tag{8a}$$

and

$$\hat{\Psi}(\mu_1, \mu_2; x, \mu) = -\frac{1}{\mu} \int_x^{\infty} \begin{vmatrix} e^{\sigma(x'-x)/\mu} & 0 \\ 0 & e^{(x'-x)/\mu} \end{vmatrix} \times \mathbf{CF}(\mu_1, \mu_2; x') dx', \quad \mu < 0, \tag{8b}$$

which can be entered into Eq. (6) to establish the integral equation

$$\mathbf{F}(\mu_1, \mu_2; x) = \int_0^{\infty} \begin{vmatrix} E_1(\sigma|x - x'|) & 0 \\ 0 & E_1(|x - x'|) \end{vmatrix} \times \mathbf{CF}(\mu_1, \mu_2; x') dx' + \mathbf{Q}(\mu_1, \mu_2; x), \tag{9}$$

where

$$\mathbf{Q}(\mu_1, \mu_2; x) = \begin{vmatrix} e^{-\sigma x/\mu_1} & 0 \\ 0 & e^{-x/\mu_2} \end{vmatrix}. \tag{10}$$

Here $E_1(x)$ is the standard exponential integral:

$$E_1(x) = \int_0^1 e^{-x/\nu} \frac{d\nu}{\nu}. \tag{11}$$

We now wish to argue that Eq. (9) admits a unique solution for all subcritical media. For the sake of notational convenience, we prefer to write Eq. (9) as

$$\mathbf{F}(\mu_1, \mu_2; x) = \mathbf{LF}(\mu_1, \mu_2; x) + \mathbf{Q}(\mu_1, \mu_2; x), \tag{12}$$

where $\mathbf{L}$ denotes the integral operator.

II. EXISTENCE THEOREM

By investigating Eq. (12), the linear integral equation for $\mathbf{F}(\mu_1, \mu_2; x)$, we find in this section a condition sufficient to ensure the existence of a solution to the singular $\mathbf{H}$-matrix equation discussed in SBK. To establish this condition, we consider

$$\mathbf{f} = \mathbf{Lf} + \mathbf{q} \tag{13}$$

in the function space $\mathcal{L}_1$ of *vector* functions with norm

$$\|\mathbf{f}\| = \max_{i=1,2} \left\{ \int_0^{\infty} |f_i(x)| dx \right\}, \tag{14}$$

where f_i, $i = 1$ and 2 denote the two elements of **f**. Note that we take Eq. (13) to be a *vector* version of the *matrix* equation given by Eq. (12).

Theorem 1: If ρ denotes the dominant eigenvalue of the nonnegative matrix $\Sigma^{-1}\mathbf{C}$ and if $2\rho < 1$, then the equation

$$\mathbf{f} = \mathbf{L}\mathbf{f} + \mathbf{q}, \tag{15}$$

with $\mathbf{q}$ in $\mathfrak{L}_1$, has a unique solution $\mathbf{f}$ in $\mathfrak{L}_1$ given by the series

$$\mathbf{f} = \sum_{n=0}^{\infty} \mathbf{L}^n\mathbf{q}. \tag{16}$$

To prove the theorem, we note that the series given by Eq. (16) converges in $\mathfrak{L}_1$ provided the spectral radius $\|\mathbf{L}\|_{\text{sp}}$ of $\mathbf{L}$, which can be computed from[10]

$$\|\mathbf{L}\|_{\text{sp}} = \lim_{n\to\infty} \|\mathbf{L}^n\|^{1/n}, \tag{17}$$

satisfies $\|\mathbf{L}\|_{\text{sp}} < 1$. Since the kernels in the matrix integral operator $\mathbf{L}$ are nonnegative, it follows that

$$|\mathbf{L}^n\mathbf{q}| \le \mathbf{L}^n|\mathbf{q}|, \tag{18}$$

where by $|\mathbf{q}|$ we mean the vector

$$|\mathbf{q}| = \begin{vmatrix} |q_1| \\ |q_2| \end{vmatrix}. \tag{19}$$

It can be shown that

$$\int_0^\infty \mathbf{L}|\mathbf{q}|(x)dx \le 2\Sigma^{-1}\mathbf{C}\int_0^\infty |\mathbf{q}(x)|dx, \tag{20}$$

and thus it follows that

$$\int_0^\infty \mathbf{L}^n|\mathbf{q}|(x)dx \le (2\Sigma^{-1}\mathbf{C})^n\int_0^\infty |\mathbf{q}(x)|dx \tag{21}$$

and, hence, that

$$\|\mathbf{L}^n\mathbf{q}\| \le 2^n\|\mathbf{q}\|[\text{max row sum of } (\Sigma^{-1}\mathbf{C})^n]. \tag{22}$$

For a nonnegative $m \times m$ matrix, the maximum of the row sums is the ∞-norm induced on the matrix when multiplying vectors $\mathbf{v}$ with norm

$$\|\mathbf{v}\|_\infty = \max_{i=1,2,\ldots,m} \{|v_i|\}, \tag{23}$$

and thus we can write

$$\|\mathbf{L}\|_{\text{sp}} \le 2 \lim_{n\to\infty}\|(\Sigma^{-1}\mathbf{C})^n\|_\infty^{1/n} = 2\|\Sigma^{-1}\mathbf{C}\|_{\text{sp}}. \tag{24}$$

Since the spectral radius of a finite-dimensional matrix is the maximum of the absolute values of the eigenvalues of the matrix and since $\Sigma^{-1}\mathbf{C}$ is nonnegative, it follows[11] that the spectral radius coincides with the dominant root ρ. The condition $2\rho < 1$, thus, guarantees that

$$\|\mathbf{L}\|_{\text{sp}} \le 2\rho < 1, \tag{25}$$

which proves that the series given by Eq. (16) converges in $\mathfrak{L}_1$. The proof of Theorem 1 is therefore established.

It is immediately apparent that our Theorem 1 is valid for the N-group version of Eq. (12). For the two-group case, we find that the condition $2\rho < 1$ can be written explicitly in the form

$$c_{11} + \sigma c_{22} + [(c_{11} + \sigma c_{22})^2 - 4\sigma C]^{1/2} < \sigma, \tag{26}$$

which is equivalent to the two conditions

$$1 - (1/\sigma)c_{11} - c_{22} > 0 \tag{27a}$$

and

$$1 - (2/\sigma)c_{11} - 2c_{22} + (4/\sigma)C > 0. \tag{27b}$$

Here the elements of $\mathbf{C}$ are denoted by c_{ij} and $C = \det\mathbf{C}$. Henceforth, we shall assume that the inequalities (27) are satisfied.

We now wish to show that $\mathbf{F}(u, v; x)$, the *matrix* solution of Eq. (12) corresponding to the inhomogeneous term $\mathbf{Q}(u, v; x)$, defines a function which satisfies the singular $\mathbf{H}$-matrix equation discussed in SBK. We thus consider

$$\mathbf{F}(u, v; x) = \mathbf{L}\mathbf{F}(u, v; x) + \mathbf{Q}(u, v; x), \tag{28}$$

with

$$\mathbf{Q}(u, v; x) = \begin{vmatrix} e^{-\sigma x/u} & 0 \\ 0 & e^{-x/v} \end{vmatrix}, \quad \text{Re}u > 0, \ \text{Re}v > 0. \tag{29}$$

In regard to Eq. (28), we now wish to establish

Theorem 2: For each value of $x \in [0, \infty)$, the function $\mathbf{F}(u, v; x)$ is analytic in u and v for $\text{Re}u > 0$ and $\text{Re}v > 0$.

To prove the theorem, we first note that we can write, subject to the conditions given by inequalities (27),

$$\mathbf{F}(u, v; x) = \sum_{n=0}^{\infty} \mathbf{F}_n(u, v; x), \tag{30}$$

where for fixed x, the function

$$\mathbf{F}_n(u, v; x) = \mathbf{L}^n\mathbf{Q}(u, v; x) \tag{31}$$

can clearly be seen to be analytic in u and v, for $\text{Re}u > 0$ and $\text{Re}v > 0$, since the first and second columns of $\mathbf{F}_n(u, v; x)$ are independent of v and u, respectively. Now if $|\mathbf{F}|$ denotes the matrix formed by replacing the elements F_{ij} of $\mathbf{F}$ by $|F_{ij}|$, then with the obvious interpretation of inequality between matrices, we can write

$$|\mathbf{F}_n| \le \mathbf{L}^n|\mathbf{Q}| \le \mathbf{L}^n\mathbf{I}, \quad \text{Re}u > 0 \text{ and } \text{Re}v > 0, \tag{32}$$

since $|\mathbf{Q}| \le \mathbf{I}$. It now follows from the definition of $\mathbf{L}$ that

$$|\mathbf{F}_n(u, v; x)| \le (2\Sigma^{-1}\mathbf{C})^n, \quad x \in [0, \infty) \tag{33}$$

for all $\text{Re}u > 0$ and $\text{Re}v > 0$.

Each entry in the matrix $\mathbf{F}(u, v; x)$ is analytic in u and v for $\text{Re}u > 0$ and $\text{Re}v > 0$, since it is the uniform limit of analytic functions, which follows by use of the norm

$$\|\mathbf{F}\| = \max_{i,j} \{|F_{ij}|\} \tag{34}$$

and the estimate

$$\left\|\mathbf{F}(u, v; x) - \sum_{n=0}^{N} \mathbf{F}_n(u, v; x)\right\| \le \sum_{\alpha=N+1}^{\infty} \|(2\Sigma^{-1}\mathbf{C})^\alpha\|. \tag{35}$$

The right-hand side of Eq. (35) is independent of u and v and goes to zero as $N \to \infty$, since inequalities (27) require that

$$\lim_{n\to\infty} \|(2\Sigma^{-1}C)^n\|^{1/n} < 1. \quad (36)$$

Having shown that $\mathbf{F}(u, v; x)$ is analytic in u and v for $\mathrm{Re}u > 0$ and $\mathrm{Re}v > 0$, we now note

Theorem 3: For $\mathrm{Re}u > 0$ ($u \notin [0,1]$) and $\mathrm{Re}v > 0$ ($v \notin [0,1]$), the first and second columns of $\mathbf{F}(u,v;x)$ can be represented by

$$\mathbf{F}_1(u;x) = \mathbf{Q}_1(u;x) - \mathbf{F}\left(u, \frac{u}{\sigma}; x\right) u \int_{-1}^{1} \mathbf{\Theta}(\mu)\mathbf{C}\frac{d\mu}{\sigma\mu - u} \times \begin{vmatrix} 1 \\ 0 \end{vmatrix} + u\int_0^1 \mathbf{F}(\sigma\mu, \mu; x)\mathbf{\Theta}(\mu)\mathbf{C}\frac{d\mu}{\sigma\mu - u}\begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad (37a)$$

and

$$\mathbf{F}_2(v;x) = \mathbf{Q}_2(v;x) - \mathbf{F}(\sigma v, v; x) v\int_{-1}^{1}\mathbf{\Theta}(\mu)\mathbf{C}\frac{d\mu}{\mu - v}\begin{vmatrix} 0 \\ 1 \end{vmatrix} + v\int_0^1 \mathbf{F}(\sigma\mu, \mu; x)\mathbf{\Theta}(\mu)\mathbf{C}\frac{d\mu}{\mu - v}\begin{vmatrix} 0 \\ 1 \end{vmatrix}. \quad (37b)$$

Here

$$\mathbf{\Theta}(\mu) = \begin{vmatrix} \theta(\mu) & 0 \\ 0 & 1 \end{vmatrix}, \quad (38)$$

with $\theta(\mu) = 1$, $\mu \in (-1/\sigma, 1/\sigma)$, $\theta(\mu) = 0$, $\mu \notin (-1/\sigma, 1/\sigma)$,

$$\mathbf{Q}_1(u,x) = e^{-\sigma x/u}\begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad \text{and} \quad \mathbf{Q}_2(v;x) = e^{-x/v}\begin{vmatrix} 0 \\ 1 \end{vmatrix}. \quad (39)$$

Note that the variable x enters Eqs. (37) only as a parameter.

To prove Theorem 3, we first operate on Eq. (28) to obtain

$$\mathbf{LF} = \mathbf{L}(\mathbf{LF}) + \mathbf{LQ}. \quad (40)$$

Some elementary analysis can now be used to deduce that

$$\mathbf{LQ}(u,v;x) = u\left[-\mathbf{Q}\left(u, \frac{u}{\sigma}; x\right)\int_{-1}^{1}\boldsymbol{\psi}(\mu)\frac{d\mu}{\sigma\mu - u} + \int_0^1 \mathbf{Q}(\sigma\mu, \mu; x)\boldsymbol{\psi}(\mu)\frac{d\mu}{\sigma\mu - u}\right]\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + v\left(-\mathbf{Q}(\sigma v, v; x)\int_{-1}^{1}\boldsymbol{\psi}(\mu)\frac{d\mu}{\mu - v} + \int_0^1 \mathbf{Q}(\sigma\mu, \mu; x)\boldsymbol{\psi}(\mu)\frac{d\mu}{\mu - v}\right)\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}, \quad (41)$$

where

$$\boldsymbol{\psi}(\mu) = \theta(\mu)\mathbf{C}. \quad (42)$$

Now since

$$\mathbf{LF} = (\mathbf{I} - \mathbf{L})^{-1}\mathbf{LQ} \quad \text{and} \quad \mathbf{F} = (\mathbf{I} - \mathbf{L})^{-1}\mathbf{Q}, \quad (43)$$

we conclude that $\mathbf{LF}$ can be expressed in terms of $\mathbf{F}$ and subsequently used in $\mathbf{F} = \mathbf{LF} + \mathbf{Q}$ to give Eqs. (37).

If we now define

$$\mathbf{F}(\sigma z, z; 0) = \tilde{\mathbf{H}}(z), \qquad \mathrm{Re}z > 0, \quad (44)$$

then clearly $\mathbf{H}(z)$ will be analytic in the half plane $\mathrm{Re}z > 0$, and from Eqs. (37) it follows that

$$\tilde{\mathbf{H}}(z)\mathbf{\Lambda}(z) = \mathbf{I} + z\int_0^1 \tilde{\mathbf{H}}(\mu)\boldsymbol{\psi}(\mu)\frac{d\mu}{\mu - z}, \qquad \mathrm{Re}z > 0, \quad (45)$$

where

$$\mathbf{\Lambda}(z) = \mathbf{I} + z\int_{-1}^{1}\boldsymbol{\psi}(\mu)\frac{d\mu}{\mu - z}. \quad (46)$$

Equation (45) clearly relates $\mathbf{H}(z)$ in the right-half plane to $\mathbf{H}(\mu)$, $\mu \in [0,1]$. Since, by Theorem 1, the existence of a unique $\mathbf{F}(\sigma z, z; x)$ has been established, and since Theorem 2 ensures that $\mathbf{F}(\sigma z, z; x)$ is analytic for $\mathrm{Re}z > 0$, it follows from Eq. (44) that there is at least one $\mathbf{H}$ matrix, analytic for $\mathrm{Re}z > 0$, which satisfies Eq. (45). It is also clear that there is at least one H matrix, analytic for $\mathrm{Re}z > 0$, which satisfies the N-group version of Eq. (45) when the conditions of Theorem 1 are satisfied.

III. THE RIEMANN PROBLEM AND PARTIAL INDICES

Equation (45) as established in the previous section can now be used to derive the system of singular integral equations discussed in SBK. Since $\mathbf{H}(z)$ is analytic for $\mathrm{Re}z > 0$, we deduce, upon invoking the Plemelj formulas[12] and Eq. (45), that

$$\tilde{\mathbf{H}}(\mu)\mathbf{\Lambda}^{\pm}(\mu) = \mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\nu)\boldsymbol{\psi}(\nu)\frac{d\nu}{\nu - \mu} \pm \pi i\mu\tilde{\mathbf{H}}(\mu)\boldsymbol{\psi}(\mu), \qquad \mu \in (0,1), \quad (47)$$

where the $+(-)$ superscript denotes the limiting value as $z \to \mu$ in the upper (lower) half-plane. We can now average the two Eqs. (47) to find

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\nu)\boldsymbol{\psi}(\nu)\frac{d\nu}{\nu - \mu}, \qquad \mu \in (0,1), \quad (48)$$

where

$$\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P\int_{-1}^{1}\boldsymbol{\psi}(\nu)\frac{d\nu}{\nu - \mu}. \quad (49)$$

Since the integral term in Eq. (48) does not involve all of the elements of $\tilde{\mathbf{H}}(\mu)$ on the interval $0 < \mu < 1$, we prefer to replace Eq. (48) by the equivalent system considered in SBK:

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\nu)\boldsymbol{\psi}(\nu)\frac{d\nu}{\nu - \mu}, \qquad \mu \in \left(0, \frac{1}{\sigma}\right), \quad (50)$$

and

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu)\mathbf{M}^{(2)}(\mu) = \left(\mathbf{I} + \mu P\int_0^1 \tilde{\mathbf{H}}(\nu)\boldsymbol{\psi}(\nu)\frac{d\nu}{\nu - \mu}\right)\mathbf{M}^{(2)}(\mu), \qquad \mu \in \left(\frac{1}{\sigma}, 1\right), \quad (51)$$

where

$$\mathbf{M}^{(2)}(\mu) \propto \begin{vmatrix} -\Lambda_{12}(\mu) \\ \Lambda_{11}(\mu) \end{vmatrix}. \quad (52)$$

If, as in SBK, we now introduce the sectionally analytic matrix

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_0^1 \tilde{\mathbf{H}}(\nu)\boldsymbol{\psi}(\nu)\frac{d\nu}{\nu - z}, \quad (53)$$

then the singular integral equations given by Eqs. (50) and (51) can be reduced to the equivalent inhomogeneous Riemann problem

$$\tilde{\mathbf{N}}^{+}(\mu) = \mathbf{G}(\mu)\tilde{\mathbf{N}}^{-}(\mu) + \tilde{\boldsymbol{\psi}}(\mu)[\tilde{\mathbf{\Lambda}}^{-}(\mu)]^{-1}, \qquad \mu \in (0,1), \quad (54)$$

where

$$\mathbf{G}(\mu) = \tilde{\mathbf{\Lambda}}^{+}(\mu)[\tilde{\mathbf{\Lambda}}^{-}(\mu)]^{-1}. \quad (55)$$

Except for proof that the partial indices of a canonical solution of the Riemann problem with the homogeneous boundary condition

$$\Phi^+(\mu) = \mathbf{G}(\mu)\Phi^-(\mu), \qquad \mu \in (0,1), \tag{56}$$

are nonnegative when both of the conditions $\det\mathbf{C} < 0$ *and* $\det\mathbf{\Lambda}(\infty) > 0$ apply, the analysis reported in SBK establishes the existence of a unique solution to the system of equations given by Eqs. (50) and (51) and the linear constraint

$$\left(\mathbf{I} + \nu_i \int_0^1 \tilde{\mathbf{H}}(\nu)\psi(\nu)\frac{d\nu}{\nu - \nu_i}\right)\mathbf{M}(\nu_i) = \mathbf{0}, \qquad i = 1, 2, \ldots, \kappa, \tag{57}$$

where $\nu_i, i = 1, 2, \ldots, \kappa$, are the zeros, with positive real (imaginary) part, of

$$\Lambda(z) = \det\mathbf{\Lambda}(z), \tag{58}$$

and

$$\mathbf{\Lambda}(\nu_i)\mathbf{M}(\nu_i) = \mathbf{0}, \qquad i = 1, 2, \ldots, \kappa. \tag{59}$$

In a similar vein, the half-range completeness theorem basic to the elementary solutions of Eq. (1) has been proved in SBK, except for the one elusive case $\det\mathbf{C} < 0$ *and* $\Lambda(\infty) > 0$. That completeness theorem also follows at once if a proof that the partial indices of $\Phi(z)$ are nonnegative can be established.

In the manner of Goh'berg and Krein,[9] we now can show at once that the results of Sec. II guarantee that the partial indices of $\Phi(z)$ are nonnegative for all choices of the basic parameters which satisfy inequalities (27). Note[2] that the conditions given by inequalities (27) ensure that all ν_i must be real, which includes the case $\det\mathbf{C} < 0$ *and* $\det\Lambda(\infty) > 0$.

The general solution (of finite degree at infinity) to the Riemann problem defined by Eq. (54) can be written as

$$\tilde{\mathbf{N}}(z) = \frac{1}{2\pi i}\Phi(z)\left(\int_0^1 \mathbf{K}(\nu)\frac{d\nu}{\nu - z} + \mathbf{P}(z)\right), \tag{60}$$

where

$$\mathbf{K}(\nu) = [\Phi^+(\nu)]^{-1}\tilde{\psi}(\nu)[\tilde{\Lambda}^-(\nu)]^{-1}, \tag{61}$$

$\mathbf{P}(z)$ is a matrix of polynomials, and $\Phi(z)$ is a canonical solution of the Riemann problem defined by Eq. (56). Without loss of generality, we consider $\Phi(z)$ to be of ordered normal form at infinity so that

$$\lim_{|z|\to\infty} \Phi(z)\begin{vmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2}\end{vmatrix} = \mathbf{A}, \tag{62}$$

where $\mathbf{A}$ is a constant nonsingular matrix, and κ_1 and $\kappa_2 \geq \kappa_1$ are the partial indices.

We note from Eq. (53) that $z\mathbf{N}(z)$ must be bounded as $|z|$ tends to infinity, and thus the proposition that $\kappa_1 \leq -1$ yields, from Eq. (60), the requirement that

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix}^T \int_0^1 \mathbf{K}(\nu)d\nu = \mathbf{0}. \tag{63}$$

Cauchy's integral theorem can now be used to represent $\Phi^{-1}(z)$, which subsequently can be evaluated at the origin to yield

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix}^T \int_0^1 \mathbf{K}(\nu)d\nu = -\begin{vmatrix} 1 \\ 0 \end{vmatrix}^T \Phi^{-1}(0). \tag{64}$$

It thus follows that Eqs. (63) and (64) imply that $\Phi(0)$ is singular, which, of course, contradicts the notion of $\Phi(z)$ being a canonical solution. We conclude, therefore, that if Eqs. (50) and (51) admit a solution, then the partial indices of $\Phi(z)$, the canonical solution of the Riemann problem defined by Eq. (56), cannot be negative; since the analysis of Sec. II, when inequalities (27) are satisfied, does establish the existence of a solution to Eqs. (50) and (51), we conclude that the partial indices must be nonnegative, when inequalities (27) are satisfied. Again, we note that inequalities (27) include the one case not resolved definitely in SBK. It is also apparent that the crucial proof that the partial indices for the matrix Riemann problem required in the half-range completeness theorem for the N-group problem can be taken as established, for these cases when Theorem 1 applies.

Finally, we should like to mention that Pahor and Suhadolc[13] have established the existence of a unique solution to Eq. (15); their proof, however, is based on conditions more restrictive than those of our Theorem 1.

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