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# The use of Riemann problems in solving a class of transcendental equations

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*Abstract.* A method of finding explicit expressions for the roots of a certain class of transcendental equations is discussed. In particular it is shown by determining a canonical solution of an associated Riemann boundary-value problem that expressions for the roots may be derived in closed form. The explicit solutions to two transcendental equations, $\tan\beta = \omega\beta$ and $\beta\tan\beta = \omega$, are discussed in detail, and additional specific results are given.

1. *Introduction.* In this paper we consider a technique which may be used to derive explicit expressions for the roots of certain transcendental equations. Such equations frequently occur in applications and serve to determine, amongst other things, the set of eigenvalues for eigenfunction expansions. It is often the case that an eigenvalue has a definite physical interpretation (the frequency of a normal mode for example) and as such, it can assume a special significance. It appears that the usual method of determining eigenvalues from transcendental equations is to employ an iteration procedure. While this may be satisfactory in most cases, it is still desirable to have an explicit expression for the eigenvalue, even if it is used only to determine an approximate initial value in an iteration scheme. The technique we discuss has further merit, however, in that the number of roots and their type (real or complex) is precisely determined, and the effects of various parameters upon which the solution may depend can be more easily observed. Since the method is not limited to establishing real solutions it can be used to advantage for transcendental equations with complex parameters, as encountered, for example, in Laplace-transformed, time-dependent problems in particle transport theory, or for transcendental equations for which the complex solutions are of interest (**1**).

Our method of solving transcendental equations is based on complex variable analysis and requires ultimately a canonical solution of a certain Riemann problem. The solution to the suitably posed Riemann problem follows immediately from the work of Muskhelishvili (**2**), and thus the crux of our method is in establishing the appropriate Riemann problem and making use of several elementary properties of the resulting solution to deduce roots of the given transcendental equation.

To demonstrate our procedure, we prefer first to devote section 2 to the two well-known transcendental equations $\tan\beta = \beta\omega$ and $\beta\tan\beta = \omega$, where $\omega$ is real. Though we certainly have not established criteria necessary to ensure the effectiveness of the method, we discuss in section 3 rather general sufficient conditions. We also comment on two further examples that are typical of those transcendental equations encountered in transport theory and radiative transfer. Although the two latter problems are of somewhat special interest, they are included to illustrate further why we believe this method will prove versatile and subsequently of wider interest.

Aside from the aesthetic appeal of closed-form solutions of transcendental equations, we believe computational advantages can be derived from such solutions—especially in regard to complex solutions for which a two-dimensional iterative approach can prove awkward. Our solutions to the explicit equations discussed here have been evaluated numerically, and the confirmation of known results (3) has proved elementary.

2. *Illustrative examples:* $\tan\beta = \omega\beta$ *and* $\beta\tan\beta = \omega$. In order to demonstrate the procedure involved it is probably more illuminating to consider a specific example. In particular we consider the equation

$$\tan\beta = \omega\beta, \tag{2·1}$$

where $\omega$ is real, which occurs for example in applications involving Sturm–Liouville systems. We note that the roots of equation (2·1) occur in equal and opposite pairs.

We first make the substitution

$$\beta = \frac{i}{\omega z} \tag{2·2}$$

in equation (2·1) and deduce that

$$\tan\left(\pm n\pi + \frac{i}{\omega z}\right) = \frac{i}{z} \quad (n = 0, 1, \ldots). \tag{2·3}$$

Consequently, by using some elementary identities we are able to replace equation (2·1) by

$$1 + \tfrac{1}{2}\omega z\left[\log\frac{z-1}{z+1} \pm 2n\pi i\right] = 0, \tag{2·4}$$

where the symbol 'log' denotes the principal branch of the log function in the plane cut from $-1$ to 1 along the real axis. Thus we will consider the equivalent problem of seeking the sequence of roots $\{z_n\}$, $n = 0, 1, \ldots$, in the cut plane, of equation (2·4). To this end we now introduce the notation

$$\Lambda_0(z) = 1 + \tfrac{1}{2}\omega z \log\left(\frac{z-1}{z+1}\right), \tag{2·5}$$

and

$$\Lambda_n(z) = \Lambda_0(z) + n\pi i\omega z, \tag{2·6}$$

and observe that these functions have the following obvious properties, which we use in the sequel,

$$\Lambda_0(z) = \Lambda_0(-z), \tag{2·7}$$

$$\Lambda_0(\infty) = 1 - \omega, \tag{2·8}$$

and

$$\Lambda_n(-z) = \Lambda_{-n}(z). \tag{2·9}$$

We first consider the case when $n = 0$, as this requires a slightly different treatment from the other cases. A straightforward application of the argument principle indicates, for $\omega > 0$, that $\Lambda_0(z)$ has two zeros in the cut plane. For $\omega < 0$, $\Lambda_0(z)$ does not have any zeros in the cut plane. We can further reason, on using equations (2·7) and (2·8) and also by examining equation (2·5) for $z = x > 1$, that the roots are real for $0 < \omega < 1$ and imaginary for $\omega > 1$. Consider now the homogeneous Riemann boundary-value problem whose coefficient is given by

$$G_0(t) = \frac{\Lambda_0^+(t)}{\Lambda_0^-(t)} \quad (0 < t < 1), \tag{2·10}$$

where, as usual, the $+$ ($-$) superscript denotes the limiting value of $\Lambda_0(z)$ as $z \to t$ along a non-tangential path in the halfplane $y > 0$ ($y < 0$), i.e.

$$\Lambda_0^{\pm}(t) = \lambda(t) \pm \tfrac{1}{2}\omega t\pi i, \tag{2·11}$$

where
$$\lambda(t) = 1 + \tfrac{1}{2}\omega t \ln\left(\frac{1-t}{1+t}\right). \tag{2·12}$$

The usual method, found in Muskhelishvili (2), of solving this Riemann problem indicates that

$$X_0^+(t) = G_0(t)\, X_0^-(t) \quad (0 < t < 1), \tag{2·13}$$

possesses a canonical solution of the form

$$X_0(z) = \frac{1}{z-1}\exp \Gamma_0(z) \quad (\omega > 0), \tag{2·14}$$

where
$$\Gamma_0(z) = \frac{1}{\pi}\int_0^1 \arg \Lambda_0^+(t)\frac{dt}{t-z}, \tag{2·15}$$

and where $\arg \Lambda_0^+(t)$ denotes the principal argument of $\Lambda_0^+(t)$. This canonical solution is clearly nonvanishing in the finite plane. Consider now the function

$$\psi(z) = \frac{\Lambda_0(z)}{X_0(-z)}. \tag{2·16}$$

On using equation (2·7), we find it is a straightforward matter to show that $\psi(z)$ is continuous across $-1 < t < 0$ and satisfies equation (2·13) on $0 < t < 1$. The function $\psi(z)$ is clearly analytic elsewhere in the finite plane and hence is a solution of the Riemann problem with boundary condition given by equation (2·13). Consequently we may write

$$\psi(z) = X_0(z)\, P(z) \tag{2·17}$$

where $P(z)$ is a polynomial. Equation (2·17) follows from the fact (2) that any solution of a given Riemann problem can be written as a product of a canonical solution and an entire function; since $\psi(z)$ is a solution of the Riemann problem and is of finite degree at infinity, the entire function here is thus a polynomial. Equations (2·16) and (2·17) yield

$$\Lambda_0(z) = X_0(z)\, X_0(-z)\, P(z) \tag{2·18a}$$

or since $X_0(z)$ is non-vanishing in the finite plane, $\Lambda_0(\infty) = 1-\omega$, and $\Lambda_0(z)$ has zeros at $z = \pm z_0$,

$$\Lambda_0(z) = -X_0(z)\,X_0(-z)\,(z^2-z_0^2)\,(1-\omega), \tag{2·18b}$$

where $z_0$ is clearly a zero of $\Lambda_0(z)$, $\omega \neq 1$. We note from equation (2·8) that for $\omega = 1$, $\Lambda_0(z)$ has only a root (double) at infinity, corresponding to $\beta_0 = 0$. On setting $z = 0$ in equation (2·18), we obtain

$$z_0 = i[(\omega-1)^{\frac{1}{2}}\,X_0(0)]^{-1} \quad (\omega > 0), \tag{2·19}$$

where, by equations (2·14) and (2·15)

$$z_0 = i(\omega-1)^{-\frac{1}{2}}\exp\left\{-\frac{1}{\pi}\int_0^1 \arg\Lambda_0^+(t)\frac{dt}{t}\right\} \quad (\omega > 0). \tag{2·20}$$

Thus it is apparent from equations (2·20) and (2·2) that there is only a positive real root of equation (2·1), less than $\frac{1}{2}\pi$, for $\omega > 1$. Finally, we find

$$\beta_0 = \pm\frac{1}{\omega}(\omega-1)^{\frac{1}{2}}\exp\left\{\frac{1}{\pi}\int_0^1 \arg\Lambda_0^+(t)\frac{dt}{t}\right\} \quad (\omega > 0). \tag{2·21}$$

We now consider the case $n \geqslant 1$. Here we define the even function

$$\Omega_n(z) = \Lambda_n(z)\,\Lambda_{-n}(z). \tag{2·22}$$

We proceed as before to solve the homogeneous Riemann problem whose coefficient is given by

$$G_n(t) = \frac{\Omega_n^+(t)}{\Omega_n^-(t)} \quad (0 < t < 1). \tag{2·23}$$

The canonical solution for this problem can be written as

$$X_n(z) = \exp\Gamma_n(z), \tag{2·24}$$

where

$$\Gamma_n(z) = \frac{1}{\pi}\int_0^1 \arg\Omega_n^+(t)\frac{dt}{t-z}. \tag{2·25}$$

The factorization of $\Omega_n(z)$ in terms of $X_n(z)$ takes the form

$$\Omega_n(z) = X_n(z)\,X_n(-z)\,(z^2-z_n^2)\,n^2\pi^2\omega^2. \tag{2·26}$$

On setting $z = 0$ in equation (2·26), we deduce that

$$z_n = i[n\pi\omega X_n(0)]^{-1}, \tag{2·27}$$

and so from equation (2·2)

$$\beta_n = \pm n\pi\exp\left\{\frac{1}{\pi}\int_0^1 \arg\Omega_n^+(t)\frac{dt}{t}\right\} \quad (n = 1, 2, \ldots, -\infty < \omega < \infty), \tag{2·28}$$

where from equations (2·6), (2·11) and (2·22)

$$\Omega_n^+(t) = [\Lambda_0^+(t)]^2 + n^2\pi^2\omega^2t^2. \tag{2·29}$$

A further example which can be treated in a similar manner is the determination of the roots of $\beta\tan\beta = \omega$, with $\omega$ real. By means of the substitution

$$\beta = i\omega z, \tag{2·30}$$

we consider the equivalent problem of determining the roots of the equation

$$z-\frac{1}{2\omega}\left\{\log\left(\frac{z-1}{z+1}\right)\pm 2in\pi\right\}=0. \tag{2·31}$$

By introducing the functions

$$\Lambda_0(z)=z\left\{z-\frac{1}{2\omega}\log\left(\frac{z-1}{z+1}\right)\right\}, \tag{2·32}$$

$$z\Lambda_n(z)=\Lambda_0(z)-\frac{i}{\omega}n\pi z, \tag{2·33}$$

and

$$\Omega_n(z)=\Lambda_n(z)\,\Lambda_{-n}(z), \tag{2·34}$$

and studying the Riemann problems associated with $\Lambda_0(z)$ and $\Omega_n(z)$, we find that the roots of $\beta\tan\beta=\omega$ are

$$\beta_0=\pm\left(\frac{\pi\omega}{2}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{\pi}\int_0^1\left[\arg\Lambda_0^+(t)+\frac{\pi}{2}\operatorname{sgn}\omega\right]\frac{dt}{t}\right\} \tag{2·35}$$

and

$$\beta_n=\pm\frac{\pi}{2}(4n^2-1)^{\frac{1}{2}}\exp\left\{-\frac{1}{\pi}\int_0^1\arg\Omega_n^+(t)\frac{dt}{t}\right\}\quad(n=1,2,\ldots,\ -\infty<\omega<\infty), \tag{2·36}$$

where

$$\Omega_n^+(t)=[\Lambda_0^+(t)]^2+n^2\pi^2\frac{t^2}{\omega^2}. \tag{2·37}$$

We note that equations (2·21), (2·28), (2·35), and (2·36) have been evaluated numerically to verify tables **4·19** and **4·20** in reference (3).

3. *General discussion.* Although it is perhaps not possible to specify all transcendental equations which can be solved explicitly by the method discussed here, we can give some sufficient conditions, apparent from the examples of section 2, for applying the technique.

First, the given transcendental equation

$$f(\beta)=0 \tag{3·1}$$

has to be such that it or an equivalent equation can be extended to the complex plane in such a way that if we write the resulting equation as

$$F(z)=0, \tag{3·2}$$

then $F(z)=F(-z)$, and $F(z)$ is analytic in the plane cut from $-a$ to $a$ along the real axis, except perhaps at infinity. Further we require that a solution exist to the Riemann problem with coefficient

$$G(t)=\frac{F^+(t)}{F^-(t)}\quad(0<t<a). \tag{3·3}$$

Sufficient conditions for the solvability of the considered Riemann problem are that $G(t)$ be continuous and non-vanishing (4). If, in addition, the factorization equation is to be evaluated at a point on the cut, then the coefficient $G(t)$ must be Hölder continuous at that point.

One of the principal features of our method is the requirement that we find an appropriate $F(z)$, the zeros of which will be solutions of a given transcendental equation, $f(\beta) = 0$. For example, the equation

$$f(\beta) = 1 - \frac{1}{\beta}\tan^{-1}\omega\beta = 0$$

discussed in section 2 clearly has an infinite number of solutions. On the other hand, by considering $F(z) = \Lambda_0(z)$ we were able to find the solutions of $f(\beta) = 0$ corresponding only to the principal branch of $\tan^{-1}x$. Subsequently upon taking, this time,

$$F(z) = \Lambda_n(z)\,\Lambda_n(-z)$$

and considering appropriate values of $n$, we were able to find the solutions of $f(\beta) = 0$ corresponding successively to the other branches of $\tan^{-1}x$. Note that, in general, $\Lambda_n(z) \neq \Lambda_n(-z)$, and thus by finding the zeros of $F(z)$ we have, in fact, found the zeros of $\Lambda_n(z)$, after correlating the $\pm$ zeros of $F(z)$ with those of $\Lambda_n(z)$ and $\Lambda_n(-z)$.

The simplest case, of course, is when the considered $F(z)$ has only one pair of zeros $\pm z_1$ in the cut plane; the case of two pairs of zeros $\pm z_1$ and $\pm z_2$ can easily be solved explicitly from the quadratic equation in $z_1^2$ (or $z_2^2$) resulting from the evaluation of the factorization equation at two convenient points in the complex plane. In general then, the method leads to an $N$th order algebraic equation where $N$ is the number of $\pm$ pairs of zeros of $F(z)$.

To summarize conditions sufficient for our procedure to be applicable, we note that the zeros, in the cut plane, of

$$\Lambda(z) = P(z) + \int_{-a}^{a} M(t)\frac{dt}{t-z}, \tag{3·4}$$

where $P(z)$ is a polynomial and the density function $M(t)$ satisfies the $H^*$ condition (2), can be expressed in closed form by considering, for example,

$$F(z) = \Lambda(z), \quad \text{if} \quad \Lambda(z) = \Lambda(-z), \tag{3·5}$$

or

$$F(z) = \Lambda(z)\,\Lambda(-z), \quad \text{if} \quad \Lambda(z) \neq \Lambda(-z). \tag{3·6}$$

It is clear from equations (3·5) and (3·6) that if the given $\Lambda(z)$ is even, then we can simply take $F(z) = \Lambda(z)$; whereas if $\Lambda(z)$ is not even, then an $F(z)$ as defined by equation (3·6) can be used.

Transcendental equations of precisely the type

$$\Lambda(z) = 0, \tag{3·7}$$

are often encountered in studies of particle transport theory. For example, analysis of the equation of gray radiative transfer (5) for anisotropic scattering yields a dispersion function of the form of equation (3·4) with $P(z)$ and $M(t)$ both being even polynomials.

We note also that studies of the scattering of polarized light lead to the transcendental equation

$$\tfrac{1}{8}c\Lambda_1(\beta)\,\Lambda_2(\beta) + [1 - c + \tfrac{3}{2}c(1-\omega)\beta^2]\,\Lambda_0(\beta) = 0, \quad \beta \notin (-1, 1), \tag{3·8}$$

where $c$ and $\omega$ are constants such that $0 \leqslant c, \omega \leqslant 1$,

$$\Lambda_\alpha(\beta) = (-1)^\alpha [1-3(1-\omega)\beta^2] + 3(1-\beta^2)\Lambda_0(\beta), \quad \alpha = 1 \text{ and } 2, \tag{3·9}$$

and
$$\Lambda_0(\beta) = 1 + \tfrac{1}{2}\omega\beta \int_{-1}^{1} \frac{dt}{t-\beta}. \tag{3·10}$$

Equation (3·8) can be written in the form of equation (3·4), though here the density function $M(t)$ is not a polynomial. The equation is typical of dispersion relations encountered in coupled transport problems and has been solved by the present method (6).

As a final demonstration of our method, we wish to mention a case for which a given $F(z)$ has more than one pair of zeros. The transcendental equation, for $\beta \notin (-1, 1)$,

$$1 + c_{11}\beta \int_{-1}^{1} \frac{dt}{t-\sigma\beta} + c_{22}\beta \int_{-1}^{1} \frac{dt}{t-\beta} + C\beta^2 \int_{-1}^{1} \frac{dt}{t-\sigma\beta} \int_{-1}^{1} \frac{dt}{t-\beta} = 0, \tag{3·11}$$

where the elements $c_{ij}$ of the $2 \times 2$ **C**-matrix are real and positive, $C = \det \mathbf{C}$, and $\sigma > 1$, is basic to the two-group model in neutron transport theory. Here there can be either one pair ($\pm\beta_1$) or two pairs ($\pm\beta_1$ and $\pm\beta_2$) of solutions depending (7) on $\sigma$ and **C**. We find our method of solving transcendental equations applicable here and thus, for the case of one pair of solutions to equation (3·11), we find

$$\beta_1 = \pm[\Omega(\infty)]^{-\frac{1}{2}} \exp\left\{-\frac{1}{\pi}\int_0^1 \arg \Omega^+(t)\frac{dt}{t}\right\}, \quad c_{22} > 2C \tanh^{-1}\frac{1}{\sigma}, \tag{3·12}$$

where
$$\Omega(\infty) = 1 - \frac{2}{\sigma}c_{11} - 2c_{22} + \frac{4}{\sigma}C \tag{3·13}$$

and
$$\Omega^+(t) = 1 + c_{11}t \ln\left|\frac{1-\sigma t}{1+\sigma t}\right| + c_{22}t \ln\left(\frac{1-t}{1+t}\right) + Ct^2\left[\ln\left|\frac{1-\sigma t}{1+\sigma t}\right| \ln\left(\frac{1-t}{1+t}\right) - \pi^2 \oplus(t)\right]$$
$$+ \pi i t\left[c_{11}\oplus(t) + c_{22} + Ct\left\{\ln\left(\frac{1-t}{1+t}\right)\oplus(t) + \ln\left|\frac{1-\sigma t}{1+\sigma t}\right|\right\}\right], \tag{3·14}$$

with
$$\oplus(t) = 1, \quad |t| < \frac{1}{\sigma},$$
$$= 0, \quad \text{otherwise}. \tag{3·15}$$

For the case of two pairs of solutions, we find

$$\beta_1 = \pm[\tfrac{1}{2}[A + (A^2 - 4\Delta^2)^{\frac{1}{2}}]]^{\frac{1}{2}}, \quad c_{22} \leqslant 2C \tanh^{-1}\frac{1}{\sigma}, \tag{3·16a}$$

and
$$\beta_2 = \pm[\tfrac{1}{2}[A - (A^2 - 4\Delta^2)^{\frac{1}{2}}]]^{\frac{1}{2}}, \quad c_{22} \leqslant 2C \tanh^{-1}\frac{1}{\sigma}, \tag{3·16b}$$

where
$$A = \frac{4\Omega(i)}{\Omega(\infty)} \exp\left\{-\frac{2}{\pi}\int_0^1 \arg \Omega^+(t)\frac{t\,dt}{t^2+1}\right\} - \frac{1}{\Omega(\infty)} \exp\left\{-\frac{2}{\pi}\int_0^1 \arg \Omega^+(t)\frac{dt}{t}\right\} - 1, \tag{3·17}$$

$$\Delta = \exp\left\{-\frac{1}{\pi}\int_0^1 \arg \Omega^+(t)\frac{dt}{t}\right\}, \tag{3·18}$$

and
$$\Omega(i) = 1 - \tfrac{1}{2}\pi c_{22} + (C\pi - 2c_{11})\tan^{-1}\frac{1}{\sigma}. \tag{3·19}$$

In conclusion, we note that although we have considered example problems defined only in terms of real parameters, the method is clearly valid for transcendental equations containing complex parameters. It follows therefore that the method may have an even greater advantage over standard iterative techniques for those problems dependent upon complex parameters and/or those problems the solutions of which are complex. Further, the method is not limited to finding the zeros of even functions $\Lambda(z)$ and, as is apparent from equation (3·4), our technique is appropriate for the rather broad class of functions which can be represented by Cauchy integrals; see, for example, (8). All of the example problems discussed here have been evaluated numerically. A Gaussian quadrature scheme was used to evaluate all integrals and, without any undue effort, accuracy to six significant figures was achieved.

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