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EXACT ANALYTICAL SOLUTIONS OF THE TRANSCENDENTAL EQUATION $\alpha \sin \zeta = \zeta$ *

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Abstract. Complex analysis is used to derive exact solutions of the transcendental equation $\alpha \sin \zeta = \zeta$, where α is an arbitrarily assigned complex number. The method involves the use of canonical solutions to suitably posed Riemann problems, and the explicit results are expressed in terms of elementary quadratures.

1. Introduction. Transcendental equations are, of course, encountered in many areas of analysis basic to mathematical physics or mechanics. Here we consider the equation

$$\alpha \sin \zeta = \zeta, \tag{1.1}$$

which is particularly important in studies of plane biharmonic functions in infinite or semi-infinite strips, as, for example, in the determination of the stress field in a thin plate in either plane strain or flexure [1], [2], [3]. In this case, (1.1) serves to define the required eigenvalues, and, although the roots of (1.1) can be computed by iteration, it appears that analytical solutions have not been reported in the literature. To our knowledge, the only significant analytical information available are asymptotic expressions [4], as, for example,

$$\zeta_n \sim (4n + 1)\frac{\pi}{2} + i\{-\ln \alpha + \ln (4n + 1)\pi\}, \qquad \alpha > 1. \tag{1.2}$$

Our purpose here is to develop exact closed-form solutions of equation (1.1), for general α. Our first step is to make the substitution

$$z = \alpha/\zeta, \tag{1.3}$$

and subsequently to consider the equivalent problem of seeking the zeros of

$$\Lambda(z) = 1 - \frac{1}{\alpha} z \sin^{-1} \frac{1}{z}, \tag{1.4}$$

in an appropriately cut plane. The branches of the arc sin function in the plane cut from -1 to 1, along the real axis, can be conveniently enumerated as

$$\sin^{-1} \frac{1}{z} = k\pi + (-1)^k \left[\frac{\pi}{2} - i \log \left[f(z) + \frac{1}{z} \right]\right], \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{1.5}$$

with $\log z$ denoting the principal branch of the log function, and

$$f(z) = \sqrt{1/z^2 - 1} \quad \text{and} \quad f(\infty) = i. \tag{1.6}$$

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Consequently, the zeros in the cut plane of the functions

$$\Lambda_k(z) = 1 - \frac{1}{\alpha} z\left[k\pi + (-1)^k \frac{\pi}{2} - i(-1)^k \log\left[f(z) + \frac{1}{z}\right]\right], \tag{1.7}$$

$$k = 0, \pm 1, \pm 2, \cdots,$$

will yield, on using (1.3), the roots of (1.1).

2. Basic analysis. Our previously reported procedure [5], [6], [7], [8] for solving a class of transcendental equations is based on the proposition that if an appropriate Riemann problem can be formulated, then the solution(s) of the considered transcendental equation can be expressed in terms of a canonical solution of that Riemann problem. From (1.7) we find the boundary values of $\Lambda_k(z)$ as z approaches the cut $[-1, 1]$ along a nontangential path from above $(+)$ and below $(-)$ to be

$$\Lambda_k^{\pm}(t) = 1 - \frac{1}{\alpha}\left[\pi t \Delta(k) - (-1)^k \frac{\pi}{2}|t| \pm i(-1)^k t C(t)\right], \tag{2.1}$$

where

$$\Delta(k) = k + (-1)^k \tag{2.2}$$

and

$$C(t) = \ln\left[f(t) + |t|^{-1}\right]. \tag{2.3}$$

Our procedure will now be to use (2.1) to formulate an appropriate Riemann problem(s). As will be evident from the ensuing analysis, there are three cases, arising in a natural manner, which we consider in turn.

Case (i). $k = 1$. The choice $k = 1$ in (1.5) yields the principal branch of the arc sin function and can be treated in an especially simple manner. We first seek to establish the number of zeros of $\Lambda_1(z)$ and thus employ the argument principle [9] in a domain bounded externally by a large circle of radius R, centered at the origin, and internally by a contour encircling $[-1, 1]$, which we shrink onto the cut. Since $\Lambda_1(\infty) = 1 - \alpha^{-1}$, we find that the change in argument of $\Lambda_1(z)$ on the large circle tends to zero as $R \to \infty$ (of course for $\alpha = 1$, $\Lambda_1(\infty) = 0$), whereas the change on the contour encircling the cut is 0 for $\alpha \in R_1^{(0)}$ and 4π for $\alpha \in R_1^{(2)}$, where $R_1^{(2)}$ and $R_1^{(0)}$ are the bounded and unbounded regions respectively, determined by the curve $\Lambda_1^+(t) = 0$, $t \in [-1, 1]$, as shown in Fig. 1. Consequently, $\Lambda_1(z)$ has no zeros for $\alpha \in R_1^{(0)}$ and two zeros for $\alpha \in R_1^{(2)}$. We now consider the Riemann problem defined by

$$X_1^+(t) = G_1(t) X_1^-(t), \qquad t \in (0, 1), \tag{2.4}$$

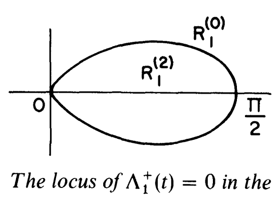

FIG. 1. *The locus of* $\Lambda_1^+(t) = 0$ *in the* α*-plane*

where

$$G_1(t) = \Lambda_1^+(t)/\Lambda_1^-(t) \tag{2.5}$$

is the Riemann coefficient and $X_1^{\pm}(t)$ denote the boundary values of a sectionally analytic function $X_1(z)$ which is nonvanishing in the finite plane. As $\Lambda_1(z) = \Lambda_1(-z)$ it is a straightforward matter to show that $\Lambda_1(z)X_1^{-1}(-z)$ is also a solution of the Riemann problem defined by (2.4) and thus [10] can be expressed in terms of the canonical solution $X_1(z)$ as

$$\Lambda_1(z)X_1^{-1}(-z) = X_1(z)P_1(z), \tag{2.6}$$

where $P_1(z)$ is a polynomial. On making use of the result for the endpoint behavior [10] of Cauchy-type integrals, namely

$$\frac{1}{2\pi i}\int_a^b \log G(t)\frac{dt}{t-z} = \mp \frac{\log G(c)}{2\pi i}\log(z-c) + \Gamma(z),$$

with the upper sign for $c = a$ and the lower sign for $c = b$, $\Gamma(z)$ being bounded at both ends, we deduce, since $\arg G_1(t)$ increases by 2π on $[0, 1]$, that the canonical solution for $\alpha \in R_1^{(2)}$ is given by

$$X_1(z) = \frac{1}{1-z}\exp\left[\frac{1}{2\pi i}\int_0^1 \log G_1(t)\frac{dt}{t-z}\right], \qquad \alpha \in R_1^{(2)}, \tag{2.7}$$

where we have chosen $\arg G_1(0) = 0$. It thus follows that (2.6) can be written as

$$\Lambda_1(z) = X_1(z)X_1(-z)[z_1^2 - z^2](1 - \alpha^{-1}), \qquad \alpha \in R_1^{(2)}, \tag{2.8}$$

where clearly $\pm z_1$ are the desired zeros of $\Lambda_1(z)$. As will be discussed in § 3, (2.8) can be evaluated at some suitable value of z to yield explicit results for $\pm z_1$.

Case (ii). *k odd, $k \neq 1$.* The simplicity of the preceding case of $k = 1$ was due to the fact that $\Lambda_1(z) = \Lambda_1(-z)$, which was essential in the development of (2.8), the *factorization* of $\Lambda_1(z)$. It is evident from (1.7) that $\Lambda_k(z)$, $k \neq 1$, is not an even function, and consequently we need to introduce an auxiliary function that can be factored in the manner of (2.8). A convenient choice is

$$\Omega_k(z) = \Lambda_k(z)\Lambda_k(-z). \tag{2.9}$$

We note from (1.7) that, with k odd,

$$\Lambda_k(-z) = \Lambda_{2-k}(z), \qquad k = 3, 5, 7, \cdots, \tag{2.10}$$

and thus we can write

$$\Omega_k(z) = \Lambda_k(z)\Lambda_{2-k}(z), \qquad k = 3, 5, 7, \cdots, \tag{2.11}$$

which means that determining the zeros of $\Omega_k(z)$, $k = 3, 5, 7, \cdots$, will simultaneously yield the zeros of both $\Lambda_k(z)$ and $\Lambda_{2-k}(z)$. Clearly, if $z_{k\alpha}$ is a zero of $\Lambda_k(z)$, then $-z_{k\alpha}$ is a zero of $\Lambda_{2-k}(z)$. If we now apply the argument principle, in the same domain as before, to $\Omega_k(z)$, $k = 3, 5, 7, \cdots$, we find that for $\alpha \in R_k^{(0)}$, $\Omega_k(z)$ has no zeros, for $\alpha \in R_k^{(2)}$, $\Omega_k(z)$ has two zeros, whereas for $\alpha \in R_k^{(4)}$, $\Omega_k(z)$ has four zeros in the cut plane. For $k = 3, 5, 7, \cdots$, the regions $R_k^{(0)}$, $R_k^{(2)}$, and $R_k^{(4)}$ are defined in the

α-plane by Fig. 2. Proceeding as for the case $k = 1$, we now consider the Riemann problems defined by

$$X_k^+(t) = G_k(t)X_k^-(t), \qquad t \in (0, 1), \tag{2.12}$$

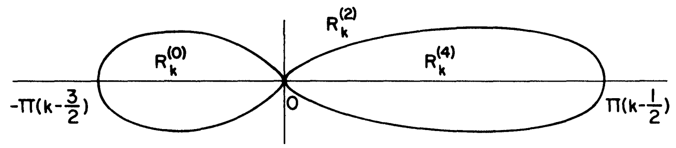

FIG. 2. *The locus of* $\Omega_k^+(t) = 0$, $k = 3, 5, 7, \cdots$, *in the* α*-plane*

with

$$G_k(t) = \Omega_k^+(t)/\Omega_k^-(t). \tag{2.13}$$

Now for $\alpha \in R_k^{(2)}$, there is no change of $\arg G_k(t)$ on $[0, 1]$, and with $\arg G_k(0) = 0$, we can write a canonical solution to (2.12) as

$$X_k(z) = \exp\left[\frac{1}{2\pi i}\int_0^1 \log G_k(t)\frac{dt}{t - z}\right], \qquad \alpha \in R_k^{(2)}. \tag{2.14}$$

It follows that $\Omega_k(z)$ can now be written as

$$\Omega_k(z) = \frac{(k-1)^2\pi^2}{\alpha^2}X_k(z)X_k(-z)[z_k^2 - z^2], \qquad \alpha \in R_k^{(2)}. \tag{2.15}$$

Now for $\alpha \in R_k^{(4)}$, $\arg G_k(t)$ increases by 2π on $[0, 1]$, and thus, again with $\arg G_k(0) = 0$, a canonical solution to (2.12) can be written as

$$X_k(z) = \frac{1}{1 - z}\exp\left[\frac{1}{2\pi i}\int_0^1 \log G_k(t)\frac{dt}{t - z}\right], \qquad \alpha \in R_k^{(4)}. \tag{2.16}$$

It follows that $\Omega_k(z)$ can now be factored as

$$\Omega_k(z) = \frac{(k-1)^2\pi^2}{\alpha^2}X_k(z)X_k(-z)[z_{k,1}^2 - z^2][z_{k,2}^2 - z^2], \qquad \alpha \in R_k^{(4)}, \tag{2.17}$$

where $\pm z_{k,1}$ and $\pm z_{k,2}$ are the desired zeros of $\Omega_k(z)$ for $k = 3, 5, 7, \cdots, \alpha \in R_k^{(4)}$.

Case (iii). *k even.* As previously, we can now apply the argument principle to

$$\Omega_k(z) = \Lambda_k(z)\Lambda_k(-z), \qquad k = 0, 2, 4, \cdots, \tag{2.18}$$

where from (1.7) we have

$$\Lambda_k(-z) = \Lambda_{-k-2}(z), \qquad k = 0, 2, 4, \cdots, \tag{2.19}$$

so that the zeros of $\Omega_k(z)$ will simultaneously yield the zeros of $\Lambda_k(z)$ and $\Lambda_{-k-2}(z)$, $k = 0, 2, 4, \cdots$. We find that for $\alpha \in R_k^{(0)}$, $\Omega_k(z)$ has no zeros, for $\alpha \in R_k^{(2)}$, $\Omega_k(z)$ has two zeros, and for $\alpha \in R_k^{(4)}$, $\Omega_k(z)$ has four zeros in the cut plane. The only difference here is that the regions $R_k^{(0)}$, $R_k^{(2)}$, and $R_k^{(4)}$ are defined somewhat differently (see

Fig. 3). We find here, for $k = 0, 2, 4, \cdots$, that $\Omega_k(z)$ can be factored as

$$\Omega_k(z) = \frac{(k+1)^2}{\alpha^2}\pi^2 X_k(z)X_k(-z)[z_k^2 - z^2], \qquad \alpha \in R_k^{(2)}, \tag{2.20}$$

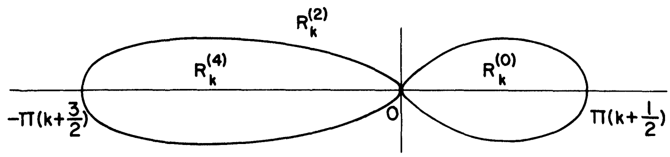

FIG. 3. *The locus of* $\Omega_k^+(t) = 0$, $k = 0, 2, 4, \cdots$, *in the* α*-plane*

and

$$\Omega_k(z) = \frac{(k+1)^2}{\alpha^2}\pi^2 X_k(z)X_k(-z)[z_{k,1}^2 - z^2][z_{k,2}^2 - z^2], \qquad \alpha \in R_k^{(4)}, \tag{2.21}$$

where the required canonical solutions are given by (2.14) and (2.16).

3. Explicit solutions. Having developed the required formalism, we are now able to solve (1.1) almost immediately. All that is required is to insert *any* convenient value(s) of z into the various factorization equations.

For the case $k = 1$, we can set $z = 0$ in (2.8) to obtain

$$z_1 = \pm(1 - \alpha^{-1})^{-1/2} \exp\left[-\frac{1}{2\pi i}\int_0^1 \log G_1(t)\frac{dt}{t}\right], \qquad \alpha \in R_1^{(2)}, \tag{3.1}$$

while the procedure of letting $|z|$ tend to infinity yields

$$z_1 = \pm\left[\frac{6\alpha - 5}{6(\alpha - 1)} - \frac{1}{\pi i}\int_0^1 t \log G_1(t)\, dt\right]^{1/2}, \qquad \alpha \in R_1^{(2)}. \tag{3.2}$$

Finally then, from (1.3) we find corresponding to (3.1),

$$\zeta_1 = \pm\alpha(1 - \alpha^{-1})^{1/2} \exp\left[\frac{1}{2\pi i}\int_0^1 \log G_1(t)\frac{dt}{t}\right], \qquad \alpha \in R_1^{(2)}, \tag{3.3a}$$

or, from (3.2),

$$\zeta_1 = \pm\alpha\left[\frac{6\alpha - 5}{6(\alpha - 1)} - \frac{1}{\pi i}\int_0^1 t \log G_1(t)\, dt\right]^{-1/2}, \qquad \alpha \in R_1^{(2)}. \tag{3.3b}$$

In a similar manner, we can set $z = 0$, or let $|z|$ tend to infinity in (2.15) and (2.20) to obtain

$$\zeta_k = \pm\Delta(k)\pi \exp\left[\frac{1}{2\pi i}\int_0^1 \log G_k(t)\frac{dt}{t}\right], \qquad \alpha \in R_k^{(2)}, \tag{3.4a}$$

or

$$\zeta_k = \pm\alpha\Delta(k)\pi\left[[\alpha + (-1)^k]^2 + i\Delta^2(k)\pi\int_0^1 t \log G_k(t)\, dt\right]^{-1/2}, \qquad \alpha \in R_k^{(2)}, \tag{3.4b}$$

which apply for $k = 3, 5, 7, \cdots$, *and* $k = 0, 2, 4, \cdots$; it must be recalled, however, that the regions $R_k^{(2)}$ are different for even or odd values of k (see Figs. 2 and 3).

The determination of $z_{k,1}$ and $z_{k,2}$ from (2.17), for $k = 3, 5, 7, \cdots$, or (2.21), for $k = 0, 2, 4, \cdots$, necessitates the evaluation of each of those two factorization equations at two distinct points, which for convenience we choose to be $z = 0$ and $z = i$. Each of the two sets of resulting equations can be solved simultaneously to yield, after (1.3) is invoked,

$$\zeta_{k,\beta} = \pm\alpha[P_k - Q_k + (-1)^\beta[(P_k - Q_k)^2 - 2Q_k]^{1/2}]^{-1/2}, \qquad \alpha \in R_k^{(4)}, \quad \beta = 1 \text{ or } 2, \tag{3.5}$$

where we have defined

$$P_k = \frac{1}{2}\left[\frac{\alpha^2\Omega_k(i)}{\Delta^2(k)\pi^2 X_k(i)X_k(-i)} - 1\right] \tag{3.6}$$

and

$$Q_k = \frac{\alpha^2}{2\Delta^2(k)\pi^2 X_k^2(0)}. \tag{3.7}$$

A solution alternative to (3.5) can be obtained by setting $z = ar$ and $z = br$, with a and b real, in (2.17) or (2.21) and subsequently letting $|r|$ tend to infinity to yield, for $k = 3, 5, 7, \cdots$, *and* $k = 0, 2, 4, \cdots$,

$$\zeta_{k,\beta} = \pm\alpha\sqrt{2}[B_k + (-1)^\beta[B_k^2 + 4C_k]^{1/2}]^{-1/2}, \qquad \alpha \in R_k^{(4)}, \quad \beta = 1 \text{ or } 2, \tag{3.8}$$

where

$$B_k = 1 - I_{k1} + \left[\frac{\alpha + (-1)^k}{\Delta(k)\pi}\right]^2, \tag{3.9}$$

$$C_k = I_{k1} - \frac{1}{2}I_{k1}^2 - I_{k3} + \frac{[\alpha(-1)^k + 1]}{3\Delta^2(k)\pi^2} + (I_{k1} - 1)\left[\frac{\alpha + (-1)^k}{\Delta(k)\pi}\right]^2, \tag{3.10}$$

and

$$I_{k\beta} = \frac{1}{\pi i}\int_0^1 t^\beta \log G_k(t)\, dt.$$

Again, the regions $R_k^{(4)}$, for which (3.5) and (3.8) are valid, are different for even or odd values of k, as shown in Figs. 2 and 3.

Equations (3.3), (3.4), (3.5) and (3.8) are our final solutions of (1.1). A Gaussian quadrature scheme has been used to evaluate our explicit solutions for numerous real and complex values of the parameter α, and solutions accurate to six significant figures were obtained quite straightforwardly.

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