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Two-Group Critical Problems for Slabs and Spheres in Neutron-Transport Theory

J. T. Kriese, C. E. Siewert, and Y. Yener

North Carolina State University
Departments of Nuclear and Mechanical Engineering
Raleigh, North Carolina 27607

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The elementary solutions of the two-group neutron-transport equation are used to solve critical problems for finite slabs and spheres. The half-range orthogonality properties of the basic eigenvectors are used, along with the fundamental $\boldsymbol{H}$-matrix, to reduce the encountered system of singular integral equations to a system of Fredholm-type equations, and these final equations are solved iteratively to yield accurate predictions of the two-group values of the extrapolated endpoint and critical dimensions for a selected set of slabs and spheres.

I. INTRODUCTION

One of the principal features of the multigroup version of the neutron-transport equation is that a reasonable degree of the fine-structure physics can be incorporated in the model, and thus the need to deal with the more general energy-dependent transport equation can often be avoided for reactor design purposes. In fact, for design analysis the multigroup diffusion-theory approximation to the transport equation is thought sufficient, in many instances, especially since the cost of performing parameter surveys based on a more exact theory can become prohibitive. Of course, many reactor calculations do require the increased accuracy obtained from more rigorous solutions of the transport equation, and, in fact, efficient diffusion-theory codes often make use of transport-theory results to define or improve the required boundary conditions.

We wish to establish here the basic analysis required to reduce two-group critical problems, for slabs and spheres, to a suitable computational form and to report our two-group predictions of the extrapolated endpoint and the critical dimensions of multiplying media with plane or spherical symmetry.

II. PROCEDURE

We first consider the two-group equation, for plane geometry, written[1] as

$$\mu \frac{\partial}{\partial x} \boldsymbol{\Psi}(x,\mu) + \boldsymbol{\Sigma}\boldsymbol{\Psi}(x,\mu) = \boldsymbol{C} \int_{-1}^{1} \boldsymbol{\Psi}(x,\mu')\, d\mu' \quad , \tag{1}$$

where, without loss of generality, we consider the transfer matrix $\boldsymbol{C}$, with nonnegative elements c_{ij}, to be symmetric (see Sec. IV). In Eq. (1), the angular-flux-vector is denoted by $\boldsymbol{\Psi}(x,\mu)$, where x is the optical variable (measured in terms of the smaller of the total cross sections, σ_1 and σ_2, of the two energy groups). In addition, μ denotes the direction cosine, as measured from the positive x-axis, of the propagating radiation. We write the $\boldsymbol{\Sigma}$-matrix as

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix} , \quad \sigma = \frac{\sigma_1}{\sigma_2} > 1 \quad . \tag{2}$$

If we now let x_0 be the half thickness of a finite slab (we take $x = 0$ to be the centerline) with multiplying properties, then we seek a solu-

[1] C. E. SIEWERT and P. S. SHIEH, *J. Nucl. Energy*, **21**, 383 (1967).

tion to Eq. (1) subject to the boundary conditions

$$\Psi(x,\mu) = \Psi(-x,-\mu), \quad \mu \in (-1,1) \tag{3a}$$

and

$$\Psi(x_0,-\mu) = \mathbf{0}, \quad \mu \in (0,1) \quad . \tag{3b}$$

Of course, the problem posed here in terms of Eq. (1) and the boundary conditions given by Eqs. (3), of establishing the critical dimension x_0, is equivalent to seeking that value of x_0 for which there exists a nontrivial symmetric, $\Phi(x) = \Phi(-x)$, solution of the homogeneous integral equation

$$\Phi(x) = \int_{-x_0}^{x_0} \begin{bmatrix} E_1(\sigma|x-x'|) & 0 \\ 0 & E_1(|x-x'|) \end{bmatrix} C\,\Phi(x')\,dx' \quad , \tag{4}$$

where the flux-vector is

$$\Phi(x) = \int_{-1}^{1} \Psi(x,\mu)\,d\mu \quad , \tag{5}$$

and $E_1(x)$ is the standard exponential integral:

$$E_1(x) = \int_0^1 \exp(-x/\nu)\,\frac{d\nu}{\nu} \quad . \tag{6}$$

For the case of a multiplying sphere, we seek a solution to

$$\mu\,\frac{\partial}{\partial r}\,\Psi(r,\mu) + \frac{(1-\mu^2)}{r}\,\frac{\partial}{\partial \mu}\,\Psi(r,\mu) + \Sigma\Psi(r,\mu) = C\int_{-1}^{1} \Psi(r,\mu')\,d\mu' \quad , \tag{7}$$

subject to the boundary condition

$$\Psi(r_0,-\mu) = \mathbf{0}, \quad \mu \in (0,1) \quad . \tag{8}$$

If we now extend the definition of

$$\Phi(r) = \int_{-1}^{1} \Psi(r,\mu)\,d\mu, \quad r \in [0,r_0] \quad , \tag{9}$$

so that

$$\Phi(-r) = \Phi(r), \quad r \in [0,r_0] \quad , \tag{10}$$

then, in the standard manner, we can show that the vector $\Phi(r)$ must satisfy

$$r\Phi(r) = \int_{-r_0}^{r_0} \begin{bmatrix} E_1(\sigma|r-r'|) & 0 \\ 0 & E_1(|r-r'|) \end{bmatrix} C\Phi(r')\,r'\,dr' \,, \tag{11}$$

after invoking Eq. (8). It therefore follows, in the manner of Mitsis,[2] that the solution to Eq. (11) can be expressed as

$$\Phi(r) = \frac{1}{r}\int_{-1}^{1} \hat{\Psi}(r,\mu)\,d\mu \quad , \tag{12}$$

where $\hat{\Psi}(r,\mu)$ must be a solution of

$$\mu\,\frac{\partial}{\partial r}\,\hat{\Psi}(r,\mu) + \Sigma\hat{\Psi}(r,\mu) = C\int_{-1}^{1} \hat{\Psi}(r,\mu')\,d\mu' \quad , \tag{13}$$

constrained to meet the boundary conditions

$$\hat{\Psi}(r,\mu) = -\hat{\Psi}(-r,-\mu), \quad \mu \in (-1,1) \quad , \tag{14a}$$

and

$$\hat{\Psi}(r_0,-\mu) = \mathbf{0}, \quad \mu \in (0,1) \quad . \tag{14b}$$

We conclude that the critical dimensions for multiplying slabs and spheres follow from the solutions of

$$\mu\,\frac{\partial}{\partial \tau}\,\Psi_\alpha(\tau,\mu) + \Sigma\Psi_\alpha(\tau,\mu) = C\int_{-1}^{1} \Psi_\alpha(\tau,\mu')\,d\mu' \quad , \quad \alpha = 1 \text{ or } 2 \quad , \tag{15}$$

subject to the boundary conditions

$$\Psi_\alpha(\tau,\mu) = -(-1)^\alpha\,\Psi_\alpha(-\tau,-\mu), \quad \mu \in (-1,1) \tag{16a}$$

and

$$\Psi_\alpha(\tau_\alpha,-\mu) = \mathbf{0}, \quad \mu \in (0,1) \quad , \tag{16b}$$

where the results for the slab are defined by $\alpha = 1$, and hence $\tau_1 = x_0$; whereas, the results for the sphere correspond to $\alpha = 2$, so that $\tau_2 = r_0$.

III. BASIC ANALYSIS

We should now like to construct the solutions to the critical problems defined, for slabs and spheres, by Eqs. (15) and (16). A general solution to Eq. (15) was reported by Siewert and Zweifel,[3] and thus we can write

$$\begin{aligned}
\Psi_\alpha(\tau,\mu) = \sum_{i=1}^{K} &[A_\alpha(\nu_i)\Phi(\nu_i,\mu)\exp(-\tau/\nu_i) \\
&+ A_\alpha(-\nu_i)\Phi(-\nu_i,\mu)\exp(\tau/\nu_i)] \\
&+ \int_0^{1/\sigma} [A_{1\alpha}^{(1)}(\nu)\Phi_1^{(1)}(\nu,\mu)\exp(-\tau/\nu) \\
&+ A_{1\alpha}^{(1)}(-\nu)\Phi_1^{(1)}(-\nu,\mu)\exp(\tau/\nu)]\,d\nu \\
&+ \int_0^{1/\sigma} [A_{2\alpha}^{(1)}(\nu)\Phi_2^{(1)}(\nu,\mu)\exp(-\tau/\nu) \\
&+ A_{2\alpha}^{(1)}(-\nu)\Phi_2^{(1)}(-\nu,\mu)\exp(\tau/\nu)]\,d\nu \\
&+ \int_{1/\sigma}^{1} [A_\alpha^{(2)}(\nu)\Phi^{(2)}(\nu,\mu)\exp(-\tau/\nu) \\
&+ A_\alpha^{(2)}(-\nu)\Phi^{(2)}(-\nu,\mu)\exp(\tau/\nu)]\,d\nu \quad ,
\end{aligned} \tag{17}$$

where the various A's are the expansion coefficients to be determined by Eq. (16). In Eq. (17), the eigenvectors $\Phi(\xi,\mu)$, $\xi = \pm\nu_i$ or $\pm\nu \in (0,1)$,

[2] G. J. MITSIS, "Transport Solutions to the Monoenergetic Critical Problems," ANL-6787, Argonne National Laboratory (1963).

[3] C. E. SIEWERT and P. F. ZWEIFEL, *Ann. Phys.*, **36**, 61 (1966).

are those given explicitly by Siewert and Zweifel[3]; since the eigenvectors have recently been relisted (in the paper hereafter referred to as SI) by Siewert and Ishiguro,[4] we do not give them here. We note that κ, in Eq. (17), denotes the number of $\pm$ pairs of zeros of $\Lambda(z) = \det \mathbf{\Lambda}(z)$ in the complex plane cut from -1 to 1 along the real axis:

$$\mathbf{\Lambda}(z) = \mathbf{I} + z\int_{-1}^{1} \boldsymbol{\psi}(\mu)\frac{d\mu}{\mu - z} \quad , \tag{18}$$

with $\mathbf{I}$ denoting the unit matrix and

$$\boldsymbol{\psi}(\mu) = \mathbf{\Theta}(\mu)\mathbf{C} = \begin{bmatrix} \theta(\mu) & 0 \\ 0 & 1 \end{bmatrix}\mathbf{C} \quad , \tag{19}$$

where $\theta(\mu) = 1$ for $\mu \in (-1/\sigma, 1/\sigma)$ and $\theta(\mu) = 0$ otherwise.

As discussed by Siewert and Shieh,[1] κ depends on σ and the elements of $\mathbf{C}$, but for the two-group model κ is either 1 or 2. In order to abbreviate the required formalism, we report here only the analysis for the case $\kappa = 1$, especially since the case $\kappa = 2$ is not significantly different. Thus restricting our attention to those values of σ and $\mathbf{C}$ which yield $\kappa = 1$, we consider multiplying media defined[1] by

$$\Lambda(\infty) = 1 - \frac{2}{\sigma}c_{11} - 2c_{22} + \frac{4}{\sigma}\det \mathbf{C} < 0 \ , \quad \kappa = 1 \ . \tag{20}$$

Since $\mathbf{\Phi}(\xi, -\mu) = \mathbf{\Phi}(-\xi, \mu)$, the solutions given by Eq. (17), for $\alpha = 1$ or 2, will inherently satisfy the required symmetry conditions, Eq. (16a), if we take, in general,

$$A_\alpha(\xi) = -(-1)^\alpha A_\alpha(-\xi) \quad , \tag{21}$$

and thus we can write (for $\kappa = 1$)

$$\begin{aligned}\mathbf{\Psi}_\alpha(\tau,\mu) = {} & B_\alpha(\nu_1)\{\mathbf{\Phi}(\nu_1,\mu)\exp[-(\tau+\tau_\alpha)/\nu_1] \\ & - (-1)^\alpha\,\mathbf{\Phi}(-\nu_1,\mu)\exp[(\tau-\tau_\alpha)/\nu_1]\} \\ & + \int_0^{1/\sigma} B_{1\alpha}^{(1)}(\nu)\{\mathbf{\Phi}_1^{(1)}(\nu,\mu)\exp[-(\tau+\tau_\alpha)/\nu] \\ & - (-1)^\alpha\,\mathbf{\Phi}_1^{(1)}(-\nu,\mu)\exp[(\tau-\tau_\alpha)/\nu]\}\,d\nu \\ & + \int_0^{1/\sigma} B_{2\alpha}^{(1)}(\nu)\{\mathbf{\Phi}_2^{(1)}(\nu,\mu)\exp[-(\tau+\tau_\alpha)/\nu] \\ & - (-1)^\alpha\,\mathbf{\Phi}_2^{(1)}(-\nu,\mu)\exp[(\tau-\tau_\alpha)/\nu]\}\,d\nu \\ & + \int_{1/\sigma}^{1} B_\alpha^{(2)}(\nu)\{\mathbf{\Phi}^{(2)}(\nu,\mu)\exp[-(\tau+\tau_\alpha)/\nu] \\ & - (-1)^\alpha\,\mathbf{\Phi}^{(2)}(-\nu,\mu)\exp[(\tau-\tau_\alpha)/\nu]\}\,d\nu \quad ,\end{aligned} \tag{22}$$

where, in general,

$$B_\alpha(\xi) = A_\alpha(\xi)\exp(\tau_\alpha/\xi) \quad . \tag{23}$$

[4]C. E. SIEWERT and Y. ISHIGURO, *J. Nucl. Energy*, **26**, 251 (1972).

The expression given by Eq. (22) is a solution of the two-group transport equation and clearly satisfies the appropriate symmetry condition. If we now impose the remaining boundary condition, Eq. (16b), then we find the expansion coefficients must be determined (to within a constant multiple) from

$$\begin{aligned}(-1)^\alpha \mathbf{F}_\alpha(\mu) = {} & B_\alpha(\nu_1)\mathbf{\Phi}(\nu_1,\mu) \\ & + \int_0^{1/\sigma}[B_{1\alpha}^{(1)}(\nu)\mathbf{\Phi}_1^{(1)}(\nu,\mu) \\ & + B_{2\alpha}^{(1)}(\nu)\mathbf{\Phi}_2^{(1)}(\nu,\mu)]\,d\nu \\ & + \int_{1/\sigma}^{1} B_\alpha^{(2)}(\nu)\mathbf{\Phi}^{(2)}(\nu,\mu)\,d\nu \ , \quad \mu \in (0,1) \quad ,\end{aligned} \tag{24}$$

where we have introduced

$$\begin{aligned}\mathbf{F}_\alpha(\mu) = {} & B_\alpha(\nu_1)\mathbf{\Phi}(-\nu_1,\mu)\exp(-2\tau_\alpha/\nu_1) \\ & + \int_0^{1/\sigma}[B_{1\alpha}^{(1)}(\nu)\mathbf{\Phi}_1^{(1)}(-\nu,\mu) \\ & + B_{2\alpha}^{(1)}(\nu)\mathbf{\Phi}_2^{(1)}(-\nu,\mu)]\exp(-2\tau_\alpha/\nu)\,d\nu \\ & + \int_{1/\sigma}^{1} B_\alpha^{(2)}(\nu)\mathbf{\Phi}^{(2)}(-\nu,\mu)\exp(-2\tau_\alpha/\nu)\,d\nu \quad .\end{aligned} \tag{25}$$

It is clear that Eq. (24) is a system of singular integral equations for the desired expansion coefficients; and, in fact, it was just this system of singular equations that was solved (only for $\alpha = 1$) numerically by Forster and Metcalf[5] to yield the critical thickness of a multiplying slab. Although modern computing facilities may render feasible iterative solutions of singular integral equations, we prefer a more traditional procedure and thus wish to reduce the given system of singular equations to a system of regular Fredholm-type equations, which will subsequently be solved iteratively. The half-range orthogonality relations reported in SI are based on a certain $\mathbf{H}$-matrix, and thus, since the existence and uniqueness proof (concerning this $\mathbf{H}$-matrix) given by Siewert, Burniston, and Kriese[6] was not limited to subcritical media, we can immediately reduce Eq. (24) to a system of nonsingular equations simply by taking inner products of Eq. (24) with the half-range adjoint basis $\mathbf{\Theta}(\xi,\mu)$ introduced in SI.

Since, as in SI,

$$\left(\frac{1}{\xi} - \frac{1}{\xi'}\right)\int_0^1 \tilde{\mathbf{\Theta}}(\xi',\mu)\mathbf{\Phi}(\xi,\mu)\mu\,d\mu = 0 \ , \quad \xi,\xi' > 0 \ , \tag{26}$$

[5]R. A. FORSTER and D. R. METCALF, *Trans. Am. Nucl. Soc.*, **12**, 637 (1969).

[6]C. E. SIEWERT, E. E. BURNISTON, and J. T. KRIESE, *J. Nucl. Energy*, in press (1972).

we can now take the inner product of Eq. (24) with each of $\boldsymbol{\Theta}(\nu_1,\mu)$, $\boldsymbol{\Theta}_\beta^{(1)}(\nu,\mu)$, $\beta = 1$ and 2, and $\boldsymbol{\Theta}^{(2)}(\nu,\mu)$, and write (SI)

$$B_\alpha(\nu_1) = (-1)^\alpha \frac{1}{N(\nu_1)} \int_0^1 \widetilde{\boldsymbol{\Theta}}(\nu_1,\mu)\boldsymbol{F}_\alpha(\mu)\mu\, d\mu \ , \quad (27a)$$

$$B_{\beta\alpha}^{(1)}(\nu) = (-1)^\alpha \frac{1}{N^{(1)}(\nu)} \int_0^1 \widetilde{\boldsymbol{\Theta}}_\beta^{(1)}(\nu,\mu)\boldsymbol{F}_\alpha(\mu)\mu\, d\mu \ , \quad (27b)$$

and

$$B_\alpha^{(2)}(\nu) = (-1)^\alpha \frac{1}{N^{(2)}(\nu)} \int_0^1 \widetilde{\boldsymbol{\Theta}}^{(2)}(\nu,\mu)\boldsymbol{F}_\alpha(\mu)\mu\, d\mu \ , \quad (27c)$$

where the normalization factors $N(\nu_1)$ and $N(\nu) = N^{(1)}(\nu)\,\theta(\nu) + N^{(2)}(\nu)\,[1 - \theta(\nu)]$ are available from SI:

$$N(\nu_1) = 2\nu_1^2 \left\{ c_{12}^2 \left[\frac{\sigma\nu_1}{(\sigma\nu_1)^2 - 1} - \tau\left(\frac{1}{\sigma\nu_1}\right)\right] + \left[c_{22} - 2C\,\nu_1\tau\left(\frac{1}{\sigma\nu_1}\right)\right]^2 \left[\frac{\nu_1}{\nu_1^2 - 1} - \tau\left(\frac{1}{\nu_1}\right)\right]\right\} , \quad (28a)$$

$$N^{(1)}(\nu) = \nu\Lambda^+(\nu)\Lambda^-(\nu) \ , \quad \nu \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right) \ , \quad (28b)$$

and

$$N^{(2)}(\nu) = \nu\Lambda^+(\nu)\Lambda^-(\nu) \ , \quad \nu \in \left(-1, -\frac{1}{\sigma}\right) \cup \left(\frac{1}{\sigma}, 1\right) \ . \quad (28c)$$

Here $\tau(x) = \operatorname{arctanh}(x)$ and $\Lambda^\pm(\nu)$ denotes the limiting values as the branch cut of $\Lambda(z)$ is approached from above (+) and below (-). The integrals in Eq. (27) have been evaluated previously (SI), and therefore with the definition

$$\boldsymbol{B}_\alpha(\nu) = \theta(\nu)\begin{bmatrix} B_{1\alpha}^{(1)}(\nu) \\ B_{2\alpha}^{(1)}(\nu)\end{bmatrix} + [1 - \theta(\nu)]\, B_\alpha^{(2)}(\nu)\boldsymbol{U}^{(2)}(\nu) \ , \quad (29)$$

and with $\boldsymbol{\Psi}(\tau,\mu)$ normalized by taking $2B_\alpha(\nu_1) = (-i)^{\alpha-1}\exp(\tau_\alpha/\nu_1)$, we can reduce Eqs. (27) to the system

$$\{1 - (-1)^\alpha \exp[-2(z_0 + \tau_\alpha)/\nu_1]\} = -2(-i)^{\alpha-1}\frac{\nu_1}{N(\nu_1)}\exp(-\tau_\alpha/\nu_1)\widetilde{\boldsymbol{U}}(\nu_1)\boldsymbol{C}\widetilde{\boldsymbol{H}}^{-1}(\nu_1)\boldsymbol{C}^{-1} \times \int_0^1 \boldsymbol{H}^{-1}(\nu')\boldsymbol{C}\boldsymbol{B}_\alpha(\nu')\exp(-2\tau_\alpha/\nu')\frac{\nu'}{\nu' + \nu_1}\, d\nu' \quad (30a)$$

and

$$\boldsymbol{B}_\alpha(\nu) = (-1)^\alpha \frac{\nu}{N(\nu)}\widetilde{\boldsymbol{W}}(\nu)\boldsymbol{C}\widetilde{\boldsymbol{H}}^{-1}(\nu)\boldsymbol{C}^{-1} \times \left[\boldsymbol{H}^{-1}(\nu_1)\boldsymbol{C}\boldsymbol{U}(\nu_1)\frac{\nu_1}{\nu_1 + \nu}\frac{(-i)^{\alpha-1}}{2}\exp(-\tau_\alpha/\nu_1) + \int_0^1 \boldsymbol{H}^{-1}(\nu')\boldsymbol{C}\boldsymbol{B}_\alpha(\nu')\exp(-2\tau_\alpha/\nu')\frac{\nu'}{\nu' + \nu}\, d\nu'\right] ,$$

$$\nu \in (0,1) \ . \quad (30b)$$

Here the $\boldsymbol{U}$ and $\boldsymbol{V}$ vectors are given by

$$\boldsymbol{U}(\nu_1) = \begin{bmatrix} -\Lambda_{12}(\nu_1) \\ \Lambda_{11}(\nu_1)\end{bmatrix} , \quad \boldsymbol{U}^{(2)}(\nu) = \begin{bmatrix} -\Lambda_{12}(\nu) \\ \Lambda_{11}(\nu)\end{bmatrix} , \quad (31a,b)$$

$$\boldsymbol{U}_1^{(1)}(\nu) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad \boldsymbol{U}_2^{(1)}(\nu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad (31c,d)$$

$$\boldsymbol{V}_1(\nu) = N_{22}(\nu)\boldsymbol{U}_1^{(1)}(\nu) - N_{12}(\nu)\boldsymbol{U}_2^{(1)}(\nu) \ , \quad (31e)$$

and

$$\boldsymbol{V}_2(\nu) = N_{11}(\nu)\boldsymbol{U}_2^{(1)}(\nu) - N_{21}(\nu)\boldsymbol{U}_1^{(1)}(\nu) \ , \quad (31f)$$

where

$$N_{11}(\nu) = 1 - 4c_{11}\,\nu\,\tau(\sigma\nu) + 4\nu^2[c_{11}^2\tau^2(\sigma\nu) + c_{12}c_{21}\tau^2(\nu)] + \pi^2\nu^2(c_{11}^2 + c_{12}c_{21}) \ , \quad (32a)$$

$$N_{ij}(\nu) = c_{ij}\,[4c_{11}\nu^2\tau^2(\sigma\nu) + 4c_{22}\nu^2\tau^2(\nu) - 2\nu\tau(\sigma\nu) - 2\nu\tau(\nu) + \pi^2\nu^2(c_{11} + c_{22})] \ , \quad i \neq j \ , \quad (32b)$$

and

$$N_{22}(\nu) = 1 - 4c_{22}\nu\tau(\nu) + 4\nu^2[c_{22}^2\tau^2(\nu) + c_{12}c_{21}\tau^2(\sigma\nu)] + \pi^2\nu^2(c_{22}^2 + c_{12}c_{21}) \ . \quad (32c)$$

In addition,

$$\boldsymbol{W}(\nu) = \theta(\nu)[\boldsymbol{V}_1(\nu)\boldsymbol{V}_2(\nu)] + [1 - \theta(\nu)]\boldsymbol{U}^{(2)}(\nu)\widetilde{\boldsymbol{U}}^{(2)}(\nu) \ , \quad (33)$$

and z_0 is the Milne-problem extrapolated endpoint (SI):

$$z_0 = -\frac{\nu_1}{2}\operatorname{Log}\left[\frac{1}{2}\frac{\nu_1}{N(\nu_1)}\widetilde{\boldsymbol{U}}(\nu_1)\boldsymbol{C}\widetilde{\boldsymbol{H}}^{-1}(\nu_1) \times \boldsymbol{C}^{-1}\boldsymbol{H}^{-1}(\nu_1)\boldsymbol{C}\boldsymbol{U}(\nu_1)\right] \ . \quad (34)$$

Although here ν_1 is strictly imaginary, we have shown that Eq. (34) yields a real z_0. The matrix $\boldsymbol{H}(\nu)$ required in Eqs. (30) is the unique solution[6] of

$$\boldsymbol{H}(\nu) = \boldsymbol{I} + \nu\boldsymbol{H}(\nu)\boldsymbol{C}\int_0^1 \widetilde{\boldsymbol{H}}(\mu)\boldsymbol{\Theta}(\mu)\frac{d\mu}{\mu + \nu} \quad (35a)$$

and

$$\left[\boldsymbol{I} + \nu_1\int_0^1 \widetilde{\boldsymbol{H}}(\mu)\boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu - \nu_1}\right]\boldsymbol{U}(\nu_1) = \boldsymbol{0} \ ; \quad (35b)$$

in addition, $\boldsymbol{H}(\nu)$ has been extended to the complex plane by

$$\widetilde{\boldsymbol{H}}(z)\boldsymbol{\Lambda}(z) = \boldsymbol{I} + z\int_0^1 \widetilde{\boldsymbol{H}}(\mu)\boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu - z} \quad . \tag{36}$$

Equation (30b) is a regular Fredholm-type equation for the required vector $\boldsymbol{B}_\alpha(\nu)$, $\nu\ \epsilon(0,1)$, and thus must now be solved subject to the critical constraint given by Eq. (30a).

IV. COMPUTATIONAL RESULTS

Since we have based our analysis on the proposition that the transfer matrix $\boldsymbol{C}$ was symmetric, we first wish to reduce the general case to that convenient form. We thus consider the two-group equations (for the slab) written as

$$\mu\frac{\partial}{\partial z}I_1(z,\mu) + \sigma_1 I_1(z,\mu) = \frac{1}{2}\sigma_{11}\int_{-1}^1 I_1(z,\mu')\,d\mu' + \frac{1}{2}\sigma_{12}\int_{-1}^1 I_2(z,\mu')\,d\mu' \tag{37a}$$

and

$$\mu\frac{\partial}{\partial z}I_2(z,\mu) + \sigma_2 I_2(z,\mu) = \frac{1}{2}\sigma_{21}\int_{-1}^1 I_1(z,\mu')\,d\mu' + \frac{1}{2}\sigma_{22}\int_{-1}^1 I_2(z,\mu')\,d\mu' \quad , \tag{37b}$$

where z is the space variable, σ_1 and σ_2 are the two total cross-sections, $I_1(z,\mu)$ and $I_2(z,\mu)$ are the angular fluxes, and the elements σ_{ij} denote scattering and group-transfer cross sections. Without loss of generality, we now take $\sigma_1 > \sigma_2$, and thus the definitions

$$\sigma = \frac{\sigma_1}{\sigma_2}\ , \quad q_{ij} = \frac{1}{2\sigma_2}\sigma_{ij}\ , \quad x = \sigma_2 z \quad , \quad \text{and } \boldsymbol{C} = \boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^{-1} \quad , \tag{38}$$

where the elements of $\boldsymbol{Q}$ are q_{ij} and

$$\boldsymbol{P} = \begin{bmatrix} (q_{21}/q_{12})^{1/2} & 0 \\ 0 & 1 \end{bmatrix} \quad , \tag{39}$$

can now be used to write Eqs. (37) as

$$\mu\frac{\partial}{\partial x}\boldsymbol{\Psi}(x,\mu) + \boldsymbol{\Sigma}\boldsymbol{\Psi}(x,\mu) = \boldsymbol{C}\int_{-1}^1 \boldsymbol{\Psi}(x,\mu')\,d\mu' \quad , \tag{40}$$

with

$$\boldsymbol{\Psi}(x,\mu) = \boldsymbol{P}\boldsymbol{I}(x,\mu) = \boldsymbol{P}\boldsymbol{I}(z\sigma_2,\mu) \quad . \tag{41}$$

Note that $\boldsymbol{C}$ is symmetric and that, naturally, the same procedure is valid for the case of spherical symmetry.

To initiate our computations, we use

$$\sigma_{11} = \sigma_{11s} + (1-\chi)\bar{\nu}_1\sigma_{1f} \quad , \tag{42a}$$

$$\sigma_{12} = \sigma_{12s} + (1-\chi)\bar{\nu}_2\sigma_{2f} \quad , \tag{42b}$$

$$\sigma_{21} = \chi\,\bar{\nu}_1\sigma_{1f} \quad , \tag{42c}$$

$$\sigma_{22} = \sigma_{22s} + \chi\bar{\nu}_2\sigma_{2f} \quad , \tag{42d}$$

for the data of Forster[7,8] given in Table I and

$$\sigma_1 = \frac{a_2}{a}\ , \quad \sigma_2 = \frac{a_1}{a} \tag{43a,b}$$

$$\sigma_{11} = \sigma_1 c_A(2\to 2) \tag{44a}$$

$$\sigma_{12} = \sigma_1 c_A(1\to 2) \tag{44b}$$

$$\sigma_{21} = \sigma_2 c_A(2\to 1) \tag{44c}$$

$$\sigma_{22} = \sigma_2 c_A(1\to 1) \tag{44d}$$

for the data sets of Boffi and Premuda[9] given in Table II to define σ and $\boldsymbol{C}$, which subsequently we consider to be the basic parameters. Of course, before we can proceed to solve Eqs. (30) numerically, we must compute the discrete eigenvalue ν_1 and the required $\boldsymbol{H}$-matrix. To obtain ν_1, we first evaluated the explicit solution given by Burniston and Siewert[10] and then refined that result by solving the transcendental equation $\Lambda(\nu_1) = 0$ iteratively. In all of our work, the standard Gaussian quadrature scheme was used to evaluate required integrals. The $\boldsymbol{H}$-matrix was obtained from (SI)

$$\boldsymbol{H}(z) = \boldsymbol{T}\boldsymbol{B}(z)\boldsymbol{L}(z)\boldsymbol{T}^{-1} \quad , \quad \kappa = 1 \quad , \tag{45}$$

after having solved

$$\boldsymbol{L}(\mu) = \boldsymbol{I} + \mu\boldsymbol{L}(\mu)\,\boldsymbol{\Delta}\int_0^1 \widetilde{\boldsymbol{L}}(\mu')\boldsymbol{R}(\mu')\frac{d\mu'}{\mu'+\mu}\ , \quad \mu\ \epsilon(0,1)\ , \quad \kappa = 1\ , \tag{46}$$

iteratively. We note that the $\boldsymbol{H}$-matrix computed in this manner satisfies both of Eqs. (35) and is thus unique. Due to the fact that here ν_1 is imaginary and $\boldsymbol{H}(\nu)$, $\nu\ \epsilon(0,1)$, is complex, all of our calculations were performed in complex mode (and in double precision) on an IBM 370/165 machine. Several numerical checks were carried out once the $\boldsymbol{H}$-matrix was established, for the selected data sets given in Tables I and II, and thus we believe our selected results given in Table III are accurate to within the usual roundoff convention.

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[7] R. A. FORSTER, PhD thesis, University of Virginia (1970).

[8] R. A. FORSTER, Private Communication (1972).

[9] V. C. BOFFI and F. PREMUDA, *Nucl. Sci. Eng.*, **38**, 205 (1969).

[10] E. E. BURNISTON and C. E. SIEWERT, *Camb. Phil. Soc.*, in press (1972).

TABLE I

Two-Group Macroscopic Cross Sections* (Refs. 7 and 8)

Case	σ_1	σ_{11s}	$\bar{\nu}_1\sigma_{1f}$	σ_{12s}	σ_2	σ_{22s}	$\bar{\nu}_2\sigma_{2f}$	χ
I	0.54628	0.42410	0.2425	0.0045552	0.33588	0.31980	0.0070425	1.0
II	2.52025	2.44383	0.12658	0.029227	0.65696	0.62568	0.002621	1.0
III	0.3456	0.26304	0.17280	0.0720	0.2160	0.078240	0.167184	0.575
IV	0.3360	0.23616	0.2503392	0.0432	0.2208	0.0792	0.29016	0.575

*In cm^{-1}.

TABLE II

Two-Group Parameters (Ref. 9)

Case	a (cm)	a_1	a_2	$c_A(1\to1)$	$c_A(1\to2)$	$c_A(2\to2)$	$c_A(2\to1)$
V	7.831	2.1	10	0.923	0.016	0.95	0.640648
VI	0.7831	0.21	1	0.923	0.016	0.95	31.5311
VII	0.07831	0.021	0.1	0.923	0.016	0.95	1917.83

TABLE III

The ***H***-Matrix for Case VII

μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	1.0	0.0	0.0	1.0
0.1	0.58837 + i0.52530	−0.19069 + i0.55479	−0.05225 + i0.40180	0.51899 + i0.42436
0.2	0.43047 + i0.44788	−0.31907 + i0.47303	−0.12609 + i0.32667	0.34976 + i0.34501
0.3	0.38138 + i0.40260	−0.35456 + i0.42521	−0.13899 + i0.28265	0.28945 + i0.29852
0.4	0.35872 + i0.37598	−0.36959 + i0.39708	−0.14091 + i0.25615	0.25821 + i0.27053
0.5	0.34594 + i0.35866	−0.37759 + i0.37879	−0.14008 + i0.23854	0.23883 + i0.25193
0.6	0.33781 + i0.34652	−0.38248 + i0.36597	−0.13852 + i0.22597	0.22551 + i0.23866
0.7	0.33222 + i0.33754	−0.38575 + i0.35649	−0.13683 + i0.21653	0.21574 + i0.22869
0.8	0.32816 + i0.33063	−0.38809 + i0.34919	−0.13521 + i0.20917	0.20824 + i0.22091
0.9	0.32508 + i0.32515	−0.38984 + i0.34340	−0.13372 + i0.20326	0.20228 + i0.21467
1.0	0.32268 + i0.32069	−0.39120 + i0.33869	−0.13237 + i0.19841	0.19742 + i0.20955

Having computed ν_1 and the ***H***-matrix, we then calculated, from Eq. (34), the Milne-problem extrapolated endpoint; these results are listed, along with ν_1, in Table IV.

Equations (30) are, as is apparent, amenable to approximation, and thus if we set $B_\alpha(\nu) = \mathbf{0}$, we find the lowest-order approximations (x_{00} and r_{00}) to the critical dimensions for the considered multiplying slabs and spheres:

$$x_{00} = \frac{1}{2}\pi|\nu_1| - z_0 \tag{47a}$$

and

$$r_{00} = \pi|\nu_1| - z_0 \quad . \tag{47b}$$

The results given by Eqs. (47) are, naturally, similar to the usual diffusion-theory expressions; they are, however, based on the transport-theory results for the discrete eigenvalue ν_1 and the Milne-problem extrapolated endpoint z_0. In fact, if we ignore the continuum coefficients in Eq. (22) we can write

$$\Phi_\alpha(\tau) = \int_{-1}^{1} \Psi_\alpha(\tau,\mu)\, d\mu$$
$$= \left[\delta_{1\alpha}\cos\frac{\tau}{|\nu_1|} + \delta_{2\alpha}\sin\frac{\tau}{|\nu_1|}\right]U(\nu_1) + \ldots , \tag{48}$$

after making use of the manner in which $B_\alpha(\nu_1)$ has been normalized. Equations (47) and (48) can now be used with Eqs. (5) and (12) to establish the lowest-order results for the flux-vectors for slabs and spheres:

$$\Phi(x) = U(\nu_1)\cos\frac{\pi x}{2(x_{00} + z_0)} + \ldots , \quad x \in [-x_{00}, x_{00}] , \tag{49a}$$

TABLE IV

The Discrete Eigenvalue and the Milne-Problem Extrapolated End Point

Case	ν_1	ν_1^{*a}	z_0
I	$i181.477$	$i181.477$	0.696309
II	$i3.54785$	$i3.54785$	0.601833
III	$i0.672584$	$i0.672584$	0.404711
IV	$i0.480216$	$i0.480216$	0.353929
V	$i1.67926$	$i1.67926$	0.537680
VI	$i0.254493$	$i0.254493$	0.181374
VII	$i0.0470444$	$i0.0470444$	0.049353

[a]From Refs. 7 and 8.

and

$$\Phi(r) = U(\nu_1)\frac{1}{r}\sin\frac{\pi r}{(r_{00}+z_0)} + \ldots , \quad r \in [0, r_{00}] \quad . \tag{49b}$$

We note that Eqs. (49) have been shown valid only for ($\kappa = 1$) the case of one pair of discrete roots. It is apparent from Eqs. (49), that the common assumption (basic to diffusion theory calculations) that the two energy groups can be extrapolated to zero with a single extrapolated endpoint has been justified here ($\kappa = 1$) with respect to the lowest-order prediction of transport theory. In the same way, the use of the extrapolated endpoint from the Milne problem has also been justified; however, a rather crucial proof that Eq. (34), for an imaginary ν_1, yields a real z_0 is required. Though, as discussed by Siewert, Burniston, and Kriese,[6] the $\boldsymbol{H}$-matrix required in Eq. (34) is complex, we have been able to show that z_0 is, in fact, real for all cases for which $\kappa = 1$.

In Table V we list, along with x_{00} and r_{00}, our "exact" values of x_0 and r_0 obtained by solving Eqs. (30) iteratively. Several numerical checks were performed once our converged solutions for τ_α and $B_\alpha(\nu)$, $\nu \in (0,1)$, were established, so that we believe the results given can be used with confidence. As suggested in a recent paper by Burniston, Mullikin, and Siewert,[11] we define, for multigroup theory, the infinite-medium multiplication factor k_{BMS} to be twice the dominant eigenvalue of $\Sigma^{-1}C$, and thus for the two-group model we can write

$$k_{BMS} = \frac{1}{\sigma}c_{11} + c_{22} + \left[\left(\frac{1}{\sigma}c_{11} + c_{22}\right)^2 - \frac{4}{\sigma}\det C\right]^{1/2} \quad . \tag{50}$$

It is clear from the results listed in Table V that the simple solutions given by Eqs. (47) yield (for the selected data sets) rather accurate values of the critical dimensions for slabs and spheres as k_{BMS} varies over a considerable range.

[11]E. E. BURNISTON, T. W. MULLIKIN, and C. E. SIEWERT, *J. Math. Phys.*, **13**, 1461 (1972).

TABLE V

Critical Dimensions

Case	k_{BMS}	Slab			Sphere	
		x_0	x_0^{*a}	x_{00}	r_0	r_{00}
I	1.0000094	284.367	284.367	284.367	569.430	569.430
II	1.0107623	4.97112	4.9711	4.97112	10.5441	10.5441
III	1.3346529	0.649377	0.649377	0.651781	1.70844	1.70827
IV	1.6372945	0.396469	0.396468	0.400392	1.15513	1.15471
V	1.0386402	2.09994	2.1000	2.10009	4.73786	4.73786
VI	1.6469082	0.210000	0.21000	0.218382	0.619651	0.618139
VII	6.4759460	0.0210000	0.021000	0.0245445	0.0992465	0.0984418

[a]From Refs. 7 and 8.