EXACT ANALYTICAL SOLUTIONS BASIC TO A CLASS OF TWO-BODY ORBITS

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Abstract. The theory of complex variables is used to establish exact analytical solutions to a class of two-body problems. In view of Lambert's theorem, two points on the conic, the chord-distance between the two points, and the time interval are considered given, and subsequently the solutions for the semi-major axis required to define the orbit are developed and expressed ultimately in terms of elementary quadratures.

1. Introduction

In this work we consider the determination of conic orbits for a class of two-body problems. In the manner of Lambert's theorem (Wintner, 1947), we take two points on the orbit and the resulting chord-distance and time-difference between the two points as given, and thus we seek an exact analytical solution for the semi-major axis required to define the orbit. In addition to the basic mathematical appeal of closed-form solutions, we believe that our established results may be of interest computationally in that they can be evaluated without recourse to iteration. In reference to Figure 1, we note that the points P_1 and P_2 are located respectively at distances r_1 and r_2 from the focus S and that c is the chord-length. As discussed by Wintner (1947), the problem is not well posed until it is specified as to whether or not the shaded area A in Figure 1 contains, in general, the focus S or, for elliptic orbits S and/or the empty focus S^*.

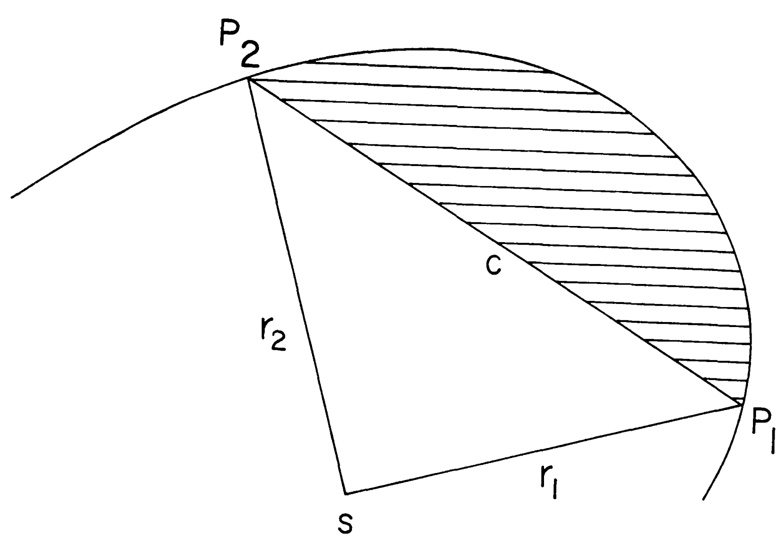

Fig. 1. The reference configuration.

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We wish to report here our exact solutions for the following cases: (i) hyperbolic or parabolic orbits with $S \notin A$, (ii) elliptic orbits with $S \notin A$ *and* $S^* \notin A$, and (iii) elliptic orbits with $S \in A$ *and* $S^* \in A$. We shall also include the possibility that the body, being first observed at P_1, has made k specified revolutions before being observed at P_2, again with the proviso that either S *and* $S^* \notin A$ or S *and* $S^* \in A$.

2. Preliminary Analysis

We consider first the hyperbolic and parabolic cases such that $S \notin A$ and those elliptic orbits with $S \notin A$ and $S^* \notin A$. For these classical cases, the problem is defined (Wintner, 1947) by

$$M = \int\limits_{(r_1+r_2-c)/2}^{(r_1+r_2+c)/2} \left[\frac{1}{x} + h\right]^{-1/2} \mathrm{d}x, \quad S \notin A, \quad \text{or} \quad S \quad \textit{and} \quad S^* \notin A. \tag{1}$$

Here r_1, r_2, and c are the distances so denoted in Figure 1, and

$$M = \sqrt{2}\,(t_2 - t_1) \tag{2}$$

is the time-difference (in appropriate units). Thus with the requirement here that $S \notin A$ or that S and $S^* \notin A$ and with M, r_1, r_2, and c given, we seek the energy constant h or, equivalently, the semi-major axis

$$a = -1/2h. \tag{3}$$

The method of solution we use here is based on the theory of complex variables and is similar to that used to solve Kepler's equation (Siewert and Burniston, 1972). We prefer first to change the integration variable in Equation (1) and to introduce

$$\Lambda(z) = M - \int\limits_{\alpha}^{\beta} \frac{1}{\sqrt{(t-z)}} \frac{\mathrm{d}t}{t^2}, \tag{4}$$

where

$$\alpha = 2(r_1 + r_2 + c)^{-1}, \tag{5a}$$

$$\beta = 2(r_1 + r_2 - c)^{-1}, \tag{5b}$$

and

$$z = -h. \tag{6}$$

We shall now restrict ourselves to the principal branch of $\sqrt{(t-z)}$, so that $-\pi < \arg(t-z) < \pi$, and thus observe that $\Lambda(z)$ is analytic in the complex plane cut from α to infinity, along the positive real axis. It therefore follows that the desired solutions will be given (except for a change of algebraic sign) by the real zeros of $\Lambda(z)$, in the plane with the cut $[\alpha, \infty)$.

We require here the limiting values of $\Lambda(z)$ as z approaches the branch cut $[\alpha, \infty)$

from above (+) and below (−); these limiting values, we find, can be expressed as

$$\Lambda^{\pm}(x) = M - \int_{x}^{\beta} \frac{1}{\sqrt{(t-x)}} \frac{\mathrm{d}t}{t^2} \mp i \int_{\alpha}^{x} \frac{1}{\sqrt{(x-t)}} \frac{\mathrm{d}t}{t^2}, \quad \alpha \leqslant x \leqslant \beta, \tag{7a}$$

and

$$\Lambda^{\pm}(x) = M \mp i \int_{\alpha}^{\beta} \frac{1}{\sqrt{(x-t)}} \frac{\mathrm{d}t}{t^2}, \quad x \geqslant \beta. \tag{7b}$$

If we now compute the change in the argument of $\Lambda(z)$ as the contour C, as given by Figure 2, (with $R \to \infty$ and $\varepsilon \to 0$) is traversed, then the argument principle (Ahlfors, 1953) reveals that $\Lambda(z)$ has no zeros in the cut plane if

$$M > \int_{\alpha}^{\beta} \frac{1}{\sqrt{(t-\alpha)}} \frac{\mathrm{d}t}{t^2} \tag{8}$$

and precisely one zero if

$$M < \int_{\alpha}^{\beta} \frac{1}{\sqrt{(t-\alpha)}} \frac{\mathrm{d}t}{t^2}. \tag{9}$$

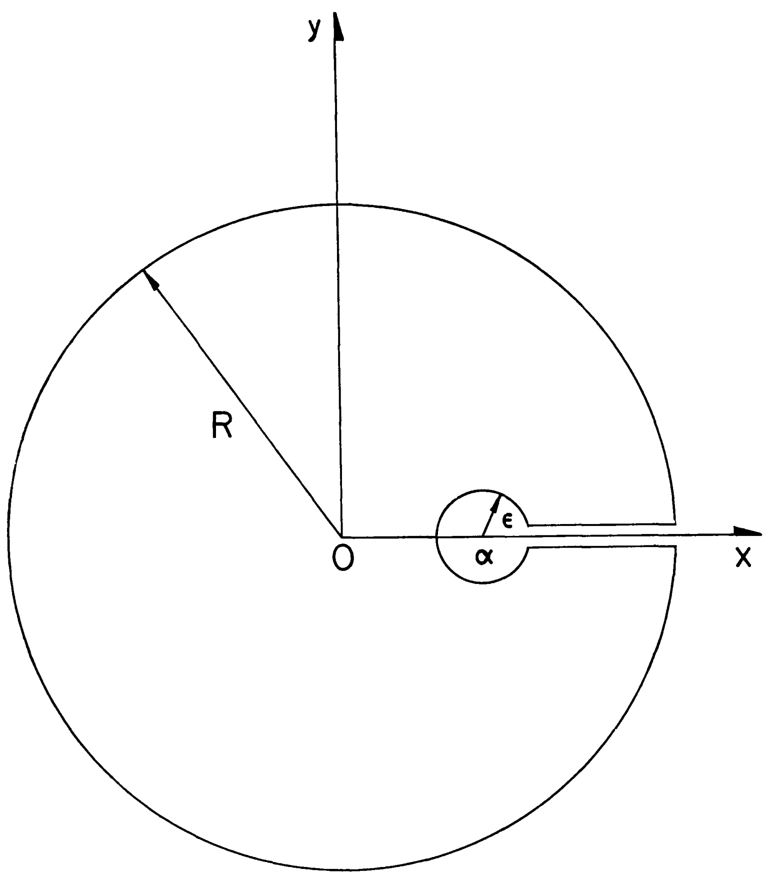

Fig. 2. The contour C.

Further, it is evident from continuity considerations that the one zero is real and can be negative only if $\Lambda(0)<0$, i.e. if

$$M < \tfrac{2}{3}(\alpha^{-3/2} - \beta^{-3/2}), \tag{10}$$

which is the hyperbolic time criterion. It thus follows from inequalities (9) and (10) that to have an elliptic orbit, with S and $S^* \notin A$, we must have

$$\tfrac{2}{3}(\alpha^{-3/2} - \beta^{-3/2}) < M < \frac{1}{\beta\sqrt{\alpha}}\left[\sqrt{\left(\frac{\beta}{\alpha} - 1\right)} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\left(\frac{\beta}{\alpha} - 1\right)}\right]. \tag{11}$$

Note that for elliptic orbits the constraint

$$M < \frac{1}{\beta\sqrt{\alpha}}\left[\sqrt{\left(\frac{\beta}{\alpha} - 1\right)} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\left(\frac{\beta}{\alpha} - 1\right)}\right] \tag{12}$$

is simply a statement that $S^* \notin A$.

If we now allow the possibility that the body upon being observed initially at position P_1 subsequently makes k complete revolutions before being observed at position P_2, again such that S *and* $S^* \notin A$, then the desired energy constant h follows, after a change of sign, from the zero (in the cut plane) of

$$\Lambda_k(z) = M - \frac{k\pi}{z\sqrt{z}} - \int_\alpha^\beta \frac{1}{\sqrt{(t-z)}}\frac{\mathrm{d}t}{t^2}, \quad S \quad \text{and} \quad S^* \notin A. \tag{13}$$

Here, in addition to the previous choice for a branch of $\sqrt{(t-z)}$, we take the principal branch of $\sqrt{z}$, so that $\Lambda_k(z)$ is analytic in the complex plane cut from $-\infty$ to 0 *and* from α to ∞. Again we can make use of the argument principle to show that $\Lambda_k(z)$ has precisely one zero in the cut plane if and only if $\Lambda_k(\alpha)>0$, i.e. if

$$M > \frac{k\pi}{\alpha\sqrt{\alpha}} + \int_\alpha^\beta \frac{1}{\sqrt{(t-\alpha)}}\frac{\mathrm{d}t}{t^2}, \quad k = 1, 2, 3, \ldots. \tag{14}$$

Further it is apparent that this zero is real and consequently must be contained in the interval $(0, \alpha)$.

The function $\Lambda_k(z)$, $k \geqslant 1$, will have two zeros if $\Lambda_k(\alpha) \leqslant 0$, but they will be complex if $|\Lambda_k(\alpha)|$ is sufficiently large. This point will be made more explicit in Section 3.

We now wish to consider the case of elliptic orbits such that both S *and* $S^* \in A$. For this situation (Wintner, 1947) Equation (1) must be replaced by

$$M = \frac{\pi}{(-h)^{3/2}} - \int_{(r_1+r_2-c)/2}^{(r_1+r_2+c)/2} \left[\frac{1}{x} + h\right]^{-1/2} \mathrm{d}x, \quad S \in A \quad \text{and} \quad S^* \in A. \tag{15}$$

Consequently, after the substitution $h=-z$, we seek a zero of

$$\hat{\Lambda}(z) = M - \frac{\pi}{z\sqrt{z}} + \int_{\alpha}^{\beta} \frac{1}{\sqrt{(t-z)}} \frac{dt}{t^2}, \quad S \text{ and } S^* \in A, \tag{16}$$

where the branches of $\sqrt{(t-z)}$ and $\sqrt{z}$ are as specified in regard to Equation (13). We note that $\hat{\Lambda}(z)$ is analytic in the plane cut from $-\infty$ to 0 and α to ∞; also, $\hat{\Lambda}(z)$ has exactly one zero in the cut plane if and only if $\hat{\Lambda}(\alpha)>0$, i.e. if

$$M > \frac{\pi}{\alpha\sqrt{\alpha}} - \int_{\alpha}^{\beta} \frac{1}{\sqrt{(t-\alpha)}} \frac{dt}{t^2}. \tag{17}$$

In a similar manner we find that if the orbit has been described k times between observations at P_1 and P_2, such that $S \in A$, and $S^* \in A$, then

$$\hat{\Lambda}_k(z) = M - \frac{(k+1)\pi}{z\sqrt{z}} + \int_{\alpha}^{\beta} \frac{1}{\sqrt{(t-z)}} \frac{dt}{t^2}, \quad S \text{ and } S^* \in A, \tag{18}$$

has exactly one zero in the plane cut from $-\infty$ to 0 and α to ∞ if and only if $\hat{\Lambda}_k(\alpha)>0$, i.e. if

$$M > \frac{(k+1)\pi}{\alpha\sqrt{\alpha}} - \int_{\alpha}^{\beta} \frac{1}{\sqrt{(t-\alpha)}} \frac{dt}{t^2}. \tag{19}$$

We shall now construct our analytical solutions for the zeros of $\Lambda(z)$, $\hat{\Lambda}(z)$, $\Lambda_b(z)$, and $\hat{\Lambda}_k(z)$, $k=1, 2, 3, \ldots$, and subsequently establish closed-form expressions for the semi-major axis a. It is evident that in as far as the analysis is concerned we may combine the cases $\hat{\Lambda}(z)$ and $\hat{\Lambda}_k(z)$, $k=1, 2, 3, \ldots$, into one case, namely $\hat{\Lambda}_k(z)$, $k=0, 1, 2, \ldots$.

3. Basic Analysis

As we have previously reported (Burniston and Siewert, 1972), a class of transcendental equations can be solved by appealing to an appropriately posed Riemann problem and subsequently factoring a given analytic function in terms of a canonical solution to the Riemann problem. In respect to the present considerations, we prefer to develop the desired solution in a manner similar to that used by Leonard (1968) to solve a problem in kinetic theory. On first considering Equation (4), we note that $\Lambda(z)$ is analytic in the plane cut from α to ∞, along the real axis, and has, in the cut plane, only one zero, say z_0, when inequality (9) is satisfied. It therefore follows that

$$F(z) = \Lambda(z)/_{(z-z_0)} \tag{20}$$

is analytic *and* non-vanishing in the same finite cut plane. If we now evaluate, from

Equation (20), the limiting values of $F(z)$, it is apparent that

$$F^+(x) = \frac{\Lambda^+(x)}{\Lambda^-(x)} F^-(x), \quad x \in (\alpha, \infty), \tag{21}$$

since $\Lambda^{\pm}(x) \neq 0$, $x \in [\alpha, \infty)$. Clearly then $F(z)$ is a canonical solution (Muskhelishvili, 1953) of the Riemann problem

$$F^+(x) = G(x) F^-(x), \quad x \in (\alpha, \infty), \tag{22}$$

where the Riemann coefficient is

$$G(x) = \Lambda^+(x)/\Lambda^-(x). \tag{23}$$

In the manner of Muskhelishvili (1953), we find it straightforward to solve Equation (22) to obtain

$$F(z) = \frac{M}{z - \alpha} \exp\left[\frac{1}{\pi} \int_\alpha^\infty \operatorname{Arg} \Lambda^+(t) \frac{\mathrm{d}t}{t - z}\right], \tag{24}$$

after imposing a normalization consistent with Equation (20). Equation (24) can now be entered into Equation (20) to yield

$$z_0 = z - \frac{\Lambda(z)(z - \alpha)}{M} \exp\left[-\frac{1}{\pi} \int_\alpha^\infty \operatorname{Arg} \Lambda^+(t) \frac{\mathrm{d}t}{t - z}\right], \tag{25}$$

where z can be assigned any convenient value. Note that $\Lambda(z)$ has been formulated such that $\operatorname{Arg}\Lambda^+(t)$ varies continuously for $t \in [\alpha, \infty)$, with $\operatorname{Arg}\Lambda^+(\alpha) = -\pi$, and that Equation (25) clearly is valid only when inequality (9) is satisfied. On using Equations (3) and (6), we can now write

$$a = \tfrac{1}{2}M \left(Mz - \Lambda(z)(z - \alpha) \exp\left[-\frac{1}{\pi} \int_\alpha^\infty \operatorname{Arg} \Lambda^+(t) \frac{\mathrm{d}t}{t - z}\right]\right)^{-1},$$

$$S \notin A \quad \text{or} \quad S, S^* \notin A. \tag{26}$$

Though the choice of z in Equation (26) can alter the computational merits of that solution, we can take $z = 0$ to yield the concise result

$$a = \frac{M}{2\alpha} [M - \tfrac{2}{3}(\alpha^{-3/2} - \beta^{-3/2})]^{-1} \exp\left[\frac{1}{\pi} \int_\alpha^\infty \operatorname{Arg} \Lambda^+(t) \frac{\mathrm{d}t}{t}\right],$$

$$S \notin A, \quad \text{or} \quad S, S^* \notin A. \tag{27}$$

For computational purposes, we view it expedient to transform the improper integral in Equation (27) to a more convenient form. The variable change $\tau = 1/t$ allows

Equation (27) to be written as

$$a = \frac{M}{2\alpha}\left[M - \tfrac{2}{3}(\alpha^{-3/2} - \beta^{-3/2})\right]^{-1} \exp\left[-\frac{1}{\pi}\int_0^{1/\beta} \tan^{-1}\left(\frac{I_3(\tau)}{M}\right)\frac{d\tau}{\tau} - \right.$$

$$\left. -\frac{1}{\pi}\int_{1/\beta}^{1/\alpha} \tan^{-1}\left(\frac{I_1(\tau)}{M - I_2(\tau)}\right)\frac{d\tau}{\tau}\right], \quad S \notin A, \quad \text{or} \quad S, S^* \notin A, \tag{28}$$

where

$$I_1(\tau) = \tau^{3/2}\left[\frac{\sqrt{(1-\alpha\tau)}}{\alpha\tau} + \tfrac{1}{2}\ln\left(\frac{1+\sqrt{(1-\alpha\tau)}}{1-\sqrt{(1-\alpha\tau)}}\right)\right], \tag{29a}$$

$$I_2(\tau) = \tau^{3/2}\left[\frac{\sqrt{(\beta\tau - 1)}}{\beta\tau} + \tan^{-1}\sqrt{(\beta\tau - 1)}\right], \tag{29b}$$

and

$$I_3(\tau) = I_1(\tau) - \tau^{3/2}\left[\frac{\sqrt{(1-\beta\tau)}}{\beta\tau} + \tfrac{1}{2}\ln\left(\frac{1+\sqrt{(1-\beta\tau)}}{1-\sqrt{(1-\beta\tau)}}\right)\right]. \tag{29c}$$

We would now like to discuss the situation when, for elliptic motion, the body has made k complete revolutions in addition to traversing the arc P_1P_2. Further, we consider that only those values of k that satisfy inequality (14) are allowed and that S and $S^* \notin A$. Here we can write

$$z\Lambda_k(z) = (z - z_k)\, F_k(z), \quad k = 1, 2, 3, \ldots, \tag{30}$$

where $F_k(z)$ is the properly normalized canonical solution of the Riemann problem

$$F_k^+(x) = G_k(x)\, F_k^-(x), \quad x \in (-\infty, 0)\, U(\alpha, \infty), \tag{31}$$

with

$$G_k(x) = \Lambda_k^+(x)/\Lambda_k^-(x). \tag{32}$$

Noting that $\operatorname{Arg}\Lambda_k^+(\pm\infty) = 0$, we can write the solution to Equation (31) as

$$F_k(z) = M\exp\left[\frac{1}{\pi}\int_{-\infty}^{0} \operatorname{Arg}\Lambda_k^+(t)\frac{dt}{t-z} + \frac{1}{\pi}\int_{\alpha}^{\infty} \operatorname{Arg}\Lambda_k^+(t)\frac{dt}{t-z}\right], \tag{33}$$

and thus obtain from Equation (30) the explicit result

$$z_k = z\left[1 - \Lambda_k(z)/F_k(z)\right], \quad S, S^* \notin A, \tag{34}$$

or

$$a_k = \frac{1}{2z}\left[1 - \frac{\Lambda_k(z)}{F_k(z)}\right]^{-1}, \quad S, S^* \notin A. \tag{35}$$

This time we elect to evaluate the explicit solution at $z = \alpha$, so that Equation (35) yields, after some elementary variable changes,

$$a_k = \frac{1}{2\alpha}\left[1 - \frac{\Lambda_k(\alpha)}{F_k(\alpha)}\right]^{-1}, \quad S, S^* \notin A, \quad k = 1, 2, 3, \ldots. \tag{36}$$

Here

$$F_k(\alpha) = M \exp\left[-\frac{1}{\pi}\int_{-1/\alpha}^{0} \tan^{-1}\left(\frac{k\pi|\tau|^{3/2}}{[1-\alpha|\tau|]^{3/2}[M - I_4(\tau)]}\right)\frac{d\tau}{\tau} - \right.$$
$$-\frac{1}{\pi}\int_{0}^{1/\beta} \tan^{-1}\left(\frac{I_3(\tau)}{M - k\pi\tau^{3/2}}\right)\frac{d\tau}{\tau(1-\alpha\tau)} -$$
$$\left. -\frac{1}{\pi}\int_{1/\beta}^{1/\alpha} \tan^{-1}\left(\frac{I_1(\tau)}{M - I_2(\tau) - k\pi\tau^{3/2}}\right)\frac{d\tau}{\tau(1-\alpha\tau)}\right] \tag{37}$$

and

$$\Lambda_k(\alpha) = M - \alpha^{-3/2}\left[k\pi + \frac{\alpha}{\beta}\sqrt{\frac{\beta}{\alpha} - 1} + \tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right], \tag{38}$$

where $I_1(\tau)$, $I_2(\tau)$, and $I_3(\tau)$ are given by Equations (29), and

$$I_4(\tau) = \frac{|\tau|^{1/2}}{\alpha[1-\alpha|\tau|]}\left[1 - \frac{\alpha}{\beta}\sqrt{1 + (\beta-\alpha)|\tau|}\right] + \frac{1}{2}\left[\frac{|\tau|}{1-\alpha|\tau|}\right]^{3/2} \times$$
$$\times\left[\ln\left(\frac{1-\sqrt{1-\alpha|\tau|}}{1+\sqrt{1-\alpha|\tau|}}\right) - \ln\left(\frac{\sqrt{1+(\beta-\alpha)|\tau|} - \sqrt{1-\alpha|\tau|}}{\sqrt{1+(\beta-\alpha)|\tau|} + \sqrt{1-\alpha|\tau|}}\right)\right]. \tag{39}$$

Since the desired results for the cases such that S *and* $S^* \in A$ follow in a manner similar to the foregoing, we simply list

$$\hat{a}_k = \frac{1}{2\alpha}\left[1 - \frac{\hat{\Lambda}_k(\alpha)}{\hat{F}_k(\alpha)}\right]^{-1}, \quad S, S^* \in A, \quad k = 0, 1, 2, \ldots, \tag{40}$$

where

$$\hat{\Lambda}_k(\alpha) = M + \alpha^{-3/2}\left[-(k+1)\pi + \frac{\alpha}{\beta}\sqrt{\frac{\beta}{\alpha} - 1} + \tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right],$$
$$k = 0, 1, 2, \ldots, \tag{41}$$

and

$$\hat{F}_k(\alpha) = M \exp\left[-\frac{1}{\pi}\int_{-1/\alpha}^{0} \tan^{-1}\left(\frac{(k+1)\pi|\tau|^{3/2}}{[1-\alpha|\tau|]^{3/2}[M + I_4(\tau)]}\right)\frac{d\tau}{\tau} - \right.$$
$$-\frac{1}{\pi}\int_{0}^{1/\beta} \tan^{-1}\left(\frac{-I_3(\tau)}{M - (k+1)\pi\tau^{3/2}}\right)\frac{d\tau}{\tau(1-\alpha\tau)} -$$
$$\left. -\frac{1}{\pi}\int_{1/\beta}^{1/\alpha} \tan^{-1}\left(\frac{-I_1(\tau)}{M + I_2(\tau) - (k+1)\pi\tau^{3/2}}\right)\frac{d\tau}{\tau(1-\alpha\tau)}\right]. \tag{42}$$

Naturally, the solutions given by Equation (40) are valid only for those values of k which satisfy inequality (19).

We finally consider the multivalued case for $\Lambda_k(z)$, S, $S^* \in A$, $k \geqslant 1$. Here, as $\Lambda_k(\alpha) < 0$, it follows that $\Lambda_k(z)$ has two zeros in the cut plane, say z_{k0} and z_{k1}, so that we may write

$$z(z-\alpha)\,\Lambda_k(z) = (z - z_{k0})\,(z - z_{k1})\,F_k(z), \tag{43}$$

where $F_k(z)$ is given by Equation (33). Thus on setting

$$K_k(z) = z(z-\alpha)\,\Lambda_k(z)\,F_k^{-1}(z), \tag{44}$$

we have

$$z_{k0} = -B(\nu,\eta) - \sqrt{B^2(\nu,\eta) - C(\nu,\eta)}, \tag{45a}$$

and

$$z_{k1} = -B(\nu,\eta) + \sqrt{B^2(\nu,\eta) - C(\nu,\eta)}, \tag{45b}$$

where

$$B(\nu,\eta) = \frac{1}{2}\left[\frac{K_k(\nu) - K_k(\eta) - \nu^2 + \eta^2}{\nu - \eta}\right], \tag{46a}$$

and

$$C(\nu,\eta) = \frac{\nu K_k(\eta) - \eta K_k(\nu) + \nu\eta(\nu-\eta)}{\nu - \eta}, \tag{46b}$$

with ν and η being two convenient points off the cut, for example $0 < \nu, \eta < \alpha$, to evaluate Equation (44). Consequently, these two zeros will lead, via Equations (3) and (6) to appropriate semi-major axes a_{k0} and a_{k1} say, if z_{k0} and z_{k1} are real. Clearly the limiting case will be when the two zeros coincide at x_k say. As this will be a double zero of $\Lambda_k(z)$ we may determine it explicitly in terms of k, α and β by considering the zeros of $\Lambda_k'(z)$. From Equation (13) we find that

$$\Lambda_k'(z) = \frac{3k\pi}{2z^{5/2}} - \frac{1}{2}\int_\alpha^\beta \frac{\mathrm{d}t}{(t-z)^{3/2}\,t^2}, \tag{47a}$$

which, after some elementary manipulation, can be written as

$$\Lambda_k'(z) = \frac{3k\pi}{2z^{5/2}} + \frac{1}{\beta z\sqrt{(\beta - z)}} - \frac{1}{\alpha z\sqrt{(\alpha - z)}} + \frac{3}{2z}\int_\alpha^\beta \frac{\mathrm{d}t}{\sqrt{(t-z)}\,t^2}. \tag{47b}$$

Now on applying the argument principle to $\Lambda_k'(z)$ in the plane cut from $-\infty$ to 0 and from α to $+\infty$ along the real axis, employing Equation (47a) for $x < 0$ and Equation (47b) for $x > \alpha$, we deduce that $\Lambda_k'(z)$ has three zeros in the finite plane of which only one will be real, namely x_k. The other two, which are not of interest here will be complex conjugates which we denote by z_k' and $\bar{z}_k'$. We may thus write

$$\Lambda_k'(z) = \frac{(z - x_k)\,(z - z_k')\,(z - \bar{z}_k')}{z^3\,(z-\alpha)\sqrt{(\beta - z)}}\,H_k(z), \tag{48}$$

where

$$H_k(z) = \frac{1}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \exp\left\{\frac{1}{\pi} \int_{-\infty}^{0} (\operatorname{Arg} \Lambda_k'^{+}(t) + \pi) \frac{dt}{t - z} + \right.$$
$$\left. + \frac{1}{\pi} \int_{\alpha}^{\beta} \operatorname{Arg} \Lambda_k'^{+}(t) \frac{dt}{t - z} + \frac{1}{\pi} \int_{\beta}^{\infty} \left(\operatorname{Arg} \Lambda_k'^{+}(t) - \frac{\pi}{2}\right) \frac{dt}{t - z}\right\}. \tag{49}$$

On setting

$$L_k(z) = z^3 (z - \alpha) \sqrt{(\beta - z)}\, \Lambda_k'(z)\, H_k^{-1}(z), \tag{50}$$

we find, on assigning z three convenient values λ, μ and ν off the cut, for example $0<\lambda$, μ, $\nu<\alpha$, that

$$(\lambda - x_k)(\lambda - z_k')(\lambda - \bar{z}_k') = L_k(\lambda), \tag{51a}$$
$$(\mu - x_k)(\mu - z_k')(\mu - \bar{z}_k') = L_k(\mu), \tag{51b}$$

and

$$(\nu - x_k)(\nu - z_k')(\nu - \bar{z}_k') = L_k(\nu), \tag{51c}$$

lead to a cubic equation for the three roots x_k, z_k' and $\bar{z}_k'$. Solving the cubic equation in the usual way for the real root x_k will yield the semi-major axis a_k:

$$a_k = \tfrac{1}{2}[A_k + B_k - \tfrac{1}{3}P_k]^{-1}, \quad S, S^* \notin A, \quad k = 1, 2, 3, \ldots, \tag{52}$$

where

$$A_k = [U_k + (U_k^2 + V_k^3)^{1/2}]^{1/3}, \tag{53a}$$
$$B_k = [U_k - (U_k^2 + V_k^3)^{1/2}]^{1/3}, \tag{53b}$$

with

$$U_k = \tfrac{1}{6}[P_k Q_k - 3R_k] - [\tfrac{1}{3}P_k]^3, \tag{53c}$$

and

$$V_k = \tfrac{1}{3}Q_k - [\tfrac{1}{3}P_k]^2. \tag{53d}$$

In addition

$$-P_k = \lambda + \mu + \nu + T\{L_k(\lambda)(\mu - \nu) + L_k(\mu)(\nu - \lambda) + L_k(\nu)(\lambda - \mu)\}, \tag{54a}$$
$$Q_k = \lambda\mu + \mu\nu + \nu\lambda + T\{L_k(\lambda)(\mu^2 - \nu^2) + L_k(\mu)(\nu^2 - \lambda^2) + L_k(\nu)(\lambda^2 - \mu^2)\}, \tag{54b}$$
$$-R_k = \lambda\mu\nu + T\{L_k(\lambda)\mu\nu(\mu - \nu) + L_k(\mu)\nu\lambda(\nu - \lambda) + L_k(\nu)\lambda\mu(\lambda - \mu)\}, \tag{54c}$$

and

$$T = [(\lambda - \mu)(\mu - \nu)(\nu - \lambda)]^{-1}. \tag{55}$$

The value of M at this point x_k, which we will denote by M_k, can easily be derived from Equation (47b) and is given by

$$M_k = \frac{2}{3}\left\{\frac{1}{\alpha\sqrt{(\alpha - x_k)}} - \frac{1}{\beta\sqrt{(\beta - x_k)}}\right\}. \tag{56}$$

Consequently the condition for two real roots in this case will be, on taking inequality (14) into account

$$\frac{2}{3}\left\{\frac{1}{\alpha\sqrt{(\alpha - x_k)}} - \frac{1}{\beta\sqrt{(\beta - x_k)}}\right\} < M < \frac{k\pi}{\alpha\sqrt{\alpha}} + \frac{1}{\beta\sqrt{\alpha}} \times$$
$$\times \left\{\sqrt{\frac{\beta}{\alpha} - 1} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right\}, \quad S, S^* \notin A, \quad k = 1, 2, 3, \ldots . \qquad (57)$$

A Gaussian quadrature scheme has been used to evaluate Equations (28), (36), (40), (45), and (52) numerically, for numerous cases, and accuracy to six significant figures was obtained rather straightforwardly.

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