Technical Notes

On a Critical Condition

C. E. Siewert and E. E. Burniston
North Carolina State University
Departments of Nuclear Engineering and Mathematics
Raleigh, North Carolina 27607
Received March 27, 1973
Revised May 4, 1973

ABSTRACT

The critical condition for a reflected spherical nuclear reactor is solved explicitly.

ANALYSIS

The critical condition for a spherical reactor, described by elementary diffusion theory, surrounded by an infinite reflector is given by

\[ B_c R \cot B_c R = 1 - \frac{D_f}{D_c} \left( 1 + \frac{R}{L_r} \right) \quad (1) \]

where

- \( R \) = radius of the sphere
- \( D_c \) = core buckling
- \( L_r \) = diffusion length in the reflector
- \( D_f \) = diffusion coefficient in the reflector
- \( D_c \) = diffusion coefficient in the core.

Equation (1) is, of course, transcendental in \( R \) and can be solved numerically by iteration; the equation can, however, be solved analytically, and the solution can thus be reduced to quadrature.

To solve Eq. (1), we first let

\[ x = B_c R, \quad \alpha = \frac{D_f}{D_c}, \text{ and } \beta = \frac{D_f}{D_c} \frac{1}{L_r B_c} \quad (2) \]

and thus seek the real solution \( x \in (0, \pi) \) of

\[ x \cot x = 1 - \alpha - \beta x, \quad \alpha > 0, \text{ and } \beta > 0 \quad (3) \]

If we now let

\[ x = \frac{\pi}{2} + \tan^{-1}\left( \frac{i}{z_0} \right) \quad (4) \]

then the value of \( z_0 \) required to complete the solution is the zero (in the cut plane) of the sectionally analytic function

\[ \Lambda(z) = lz + i \frac{\pi}{2} + \frac{1}{2} (i \beta z + 1) \int_{-1}^{1} \frac{d\mu}{\mu - z} \quad (5) \]

where

\[ l = 1 - \alpha - \beta \frac{\pi}{2} \quad (6) \]

We note that \( \Lambda(z) \) is analytic in the complex plane cut from \(-1\) to \(1\) along the real axis, that \( \Lambda(z) \) has only one zero (purely imaginary) in the cut plane, and that the boundary values of \( \Lambda(z) \) as \( z \) approaches the cut from above (+) and below (−) are given by

\[ \Lambda^+(t) = \left( \frac{l}{l - \tanh^{-1} t} + \beta \frac{\pi}{2} \right) + i \left( \frac{\pi}{2} - \beta t \tanh^{-1} t \right), \text{ for } t \in (-1, 1) \quad (7) \]

Following some basic analysis similar to that previously discussed,\(^3\) we find that we can factor \( \Lambda(z) \Lambda(-z) \) in the manner

\[ \Lambda(z) \Lambda(-z) = l^2(z_0^2 - z^2) \left( \frac{z + 1}{z - 1} \right) \exp \left[ \frac{1}{2} \int_{-1}^{1} f(t) \frac{dt}{t - z} \right] \quad (8) \]

where

\[ f(t) = \frac{\tan^{-1} \left( \frac{\pi(1 + \beta^2 t^2) \tanh^{-1} t + \pi(\alpha - 1)t}{2} \right)}{\left( \frac{\beta}{2} t \right)^2 - (l - \tanh^{-1} t)^2 - (\beta t \tanh^{-1} t)^2 + \beta \alpha \tanh^{-1} t} \]

is continuous except at \( t = 0 \), where \( f(0^-) = 3 \frac{\pi}{2} \) and \( f(0^+) = \frac{\pi}{2} \). Equation (8) is an identity in the \( z \) plane and can be solved to yield the required value of \( z_0^2 \):

\[ z_0^2 = z^2 + \frac{1}{2} \int_{-1}^{1} \left( \frac{z + 1}{z - 1} \right) \Lambda(z) \Lambda(-z) \exp \left( \frac{1}{2} \int_{-1}^{1} f(t) \frac{dt}{t - z} \right) \quad (9) \]

Although any value of \( z \) can be used in Eq. (10) to yield \( z_0^2 \), we prefer the form resulting from observing the limit as \( |z| \to \infty \):

\[ z_0^2 = 1 - J_1 - \frac{1}{72} \left( \frac{\beta^2}{2} + 2(\alpha - 1) \right) \quad (11) \]

where

\[ J_1 = \frac{2}{\pi} \int_{0}^{1} f(t) dt \quad (12) \]

We thus have reduced the desired solution to quadrature:

\[ x = \frac{\pi}{2} - \text{sgn}(l) \cot^{-1} \left| 1 - \frac{2}{72} \int_{0}^{1} f(t) dt \right| - \frac{1}{4} \left( \frac{\pi^2}{2} + \beta^2 + 2(\alpha - 1) \right)^{1/2} \quad (13) \]

The solution given by Eq. (13) has been evaluated numerically by using a Gaussian quadrature scheme to integrate \( f(t) \); quite straightforward accuracy to six significant figures was achieved for the numerous cases considered.
