Technical Notes

On a Critical Condition

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ABSTRACT

The critical condition for a reflected spherical nuclear reactor is solved explicitly.

ANALYSIS

The critical condition for a spherical reactor, described by elementary diffusion theory, surrounded by an infinite reflector is given by[1]

$$B_c R \cot B_c R = 1 - \frac{D_r}{D_c}\left(1 + \frac{R}{L_r}\right) , \qquad (1)$$

where

R = radius of the sphere

B_c^2 = core buckling

L_r = diffusion length in the reflector

D_r = diffusion coefficient in the reflector

D_c = diffusion coefficient in the core.

Equation (1) is, of course, transcendental in R and can be solved numerically by iteration; the equation can, however, be solved analytically, and the solution can thus be reduced to quadrature.

To solve Eq. (1), we first let

$$x = B_c R, \quad \alpha = \frac{D_r}{D_c}, \quad \text{and} \quad \beta = \frac{D_r}{D_c}\frac{1}{L_r B_c} , \qquad (2)$$

and thus seek the real solution $x \in (0, \pi)$ of

$$x \cot x = 1 - \alpha - \beta x , \quad \alpha > 0, \quad \text{and} \quad \beta > 0 . \qquad (3)$$

If we now let

$$x = \frac{\pi}{2} + \tan^{-1}\left(\frac{i}{z_0}\right) , \qquad (4)$$

then the value of z_0 required to complete the solution is the zero (in the cut plane) of the sectionally analytic function

$$\Lambda(z) = lz + i\frac{\pi}{2} + \frac{1}{2}(i\beta z + 1)\int_{-1}^{1}\frac{d\mu}{\mu - z} , \qquad (5)$$

where

$$l = 1 - \alpha - \beta\frac{\pi}{2} . \qquad (6)$$

We note that $\Lambda(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis, that $\Lambda(z)$ has only one zero (purely imaginary) in the cut plane, and that the boundary values of $\Lambda(z)$ as z approaches the cut from above (+) and below (-) are given by

$$\Lambda^{\pm}(t) = \left(lt - \tanh^{-1} t \mp \beta\frac{\pi}{2}t\right) + i\left(\frac{\pi}{2} \pm \frac{\pi}{2} - \beta t \tanh^{-1} t\right) , \quad t \in (-1, 1) . \qquad (7)$$

Following some basic analysis similar to that previously discussed,[2] we find that we can factor $\Lambda(z)\,\Lambda(-z)$ in the manner

$$\Lambda(z)\,\Lambda(-z) = l^2(z_0^2 - z^2)\left(\frac{z+1}{z-1}\right)\exp\left[\frac{1}{\pi}\int_{-1}^{1} f(t)\frac{dt}{t - z}\right] , \qquad (8)$$

where

$$f(t) = \tan^{-1}\left[\frac{\pi(1 + \beta^2 t^2)\tanh^{-1} t + \pi(\alpha - 1)t}{\left(\beta\frac{\pi}{2}t\right)^2 - (lt - \tanh^{-1} t)^2 - (\beta t \tanh^{-1} t)^2 + \beta\pi t \tanh^{-1} t}\right] \qquad (9)$$

is continuous except at $t = 0$, where $f(0^-) = 3\frac{\pi}{2}$ and $f(0^+) = \frac{\pi}{2}$. Equation (8) is an identity in the z plane and can be solved to yield the required value of z_0^2:

$$z_0^2 = z^2 + \frac{1}{l^2}\left(\frac{z-1}{z+1}\right)\Lambda(z)\,\Lambda(-z)\exp\left[-\frac{1}{\pi}\int_{-1}^{1} f(t)\frac{dt}{t - z}\right] . \qquad (10)$$

Although any value of z can be used in Eq. (10) to yield z_0^2, we prefer the form resulting from observing the limit as $|z| \to \infty$:

$$z_0^2 = 1 - J_1 - \frac{1}{l^2}\left[\left(\frac{\pi}{2}\right)^2 + \beta^2 + 2(\alpha - 1)\right] , \qquad (11)$$

where

$$J_1 = \frac{2}{\pi}\int_0^1 f(t)\,t\,dt . \qquad (12)$$

We thus have reduced the desired solution to quadrature:

$$x = \frac{\pi}{2} - \mathrm{sgn}(l)\cot^{-1}\left|1 - \frac{2}{\pi}\int_0^1 f(t)\,t\,dt - \frac{1}{l^2}\left[\left(\frac{\pi}{2}\right)^2 + \beta^2 + 2(\alpha - 1)\right]\right|^{1/2} . \qquad (13)$$

The solution given by Eq. (13) has been evaluated numerically by using a Gaussian quadrature scheme to integrate $f(t)$; quite straightforward accuracy to six significant figures was achieved for the numerous cases considered.

[1]R. L. MURRAY, *Nuclear Reactor Physics*, p. 68, Prentice Hall, Englewood Cliffs, New Jersey (1957).

[2]E. E. BURNISTON and C. E. SIEWERT, *Proc. Camb. Phil. Soc.*, **73**, 111 (1973).