Discrete spectrum basic to kinetic theory

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The theory of complex variables is used to establish explicit solutions to the transcendental equation that defines the discrete spectrum for a model problem in the kinetic theory of gases.

The time-dependent BGK model in the kinetic theory of gases can be linearized and expressed in the form

\[
\left( \frac{\partial}{\partial t} + c_s \frac{\partial}{\partial x} + 1 \right) h(x, c, t) = \pi^{-3/2} \int h(x, c', t) \exp(-c'^3) \, dc',
\]

(1)

where \( h(x, c, t) \) represents the perturbation of the distribution function from the Maxwellian distribution, and velocity, time, and space variables. In the manner of Cercignani, we find that Eq. (1) can be decomposed, by taking appropriate moments, into a set of two coupled integrodifferential equations plus three uncoupled equations. Since the uncoupled equations have been discussed in considerable detail, we wish to report some analysis basic to the set of coupled equations:

\[
\left( \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x} + 1 \right) \Psi(x, \mu, t) = \pi^{-3/2} \int_{-\infty}^{+\infty} Q(\mu)^{1/2} \Phi(\mu) \exp(-\mu^2) \, d\mu.
\]

(2)

Here, the elements of the two-vector \( \Psi(x, \mu, t) \) are related to the density and temperature of the gas, \( x, \mu, \) and \( t \) represent, respectively, the position, velocity component, and time, in dimensionless units. In addition:

\[
Q(\mu) = \begin{bmatrix}
(\frac{3}{2})^{1/2} (\mu^2 - \frac{1}{2}) & 1 \\
(\frac{3}{2})^{1/2} & 0
\end{bmatrix} \quad P(\mu) = (2)^{1/2} \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}.
\]

Letting

\[
\Psi(x, \mu, t) = \exp(s_1 \Phi(x, \mu, t) \exp(\nu(s_1 + 1) x / \nu),
\]

where \( s \) is complex, but \( s \neq -1 \), and \( \nu \) is to be determined, we find that the discrete spectrum consists of the zeros \( \nu(s) \) of \( \Lambda(s; s) = \det \Lambda(s; s) \), where

\[
\Lambda(s; s) = 1 + 2 \int_{-\infty}^{+\infty} \Psi(\xi; s) \frac{d\xi}{\xi - s},
\]

(3)

\[
\Psi(\xi; s) = \int_{1/(s+1)^{3/2}}^{1/(s+1)^{2/2}} \Phi(\xi) Q(\xi) \exp(-\xi^3),
\]

and

\[
T = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

It is apparent that the matrix \( \Lambda(s; s) \) is analytic in the \( s \) plane cut from \(-\infty \) to \(+\infty \) along the real axis.

In this note, we develop explicit expressions for the zeros \( \nu(\xi) \) of \( \Lambda(\xi; s) \). The analysis is based on the classical work of Mushkelishvili and makes use of the method we recently reported for solving a class of transcendental equations. Upon expanding \( \Lambda(\xi; s) \), we find

\[
\Lambda(\xi; s) = \int_{-\infty}^{+\infty} \exp(-\xi z) \frac{d\xi}{\xi - s},
\]

(4)

We first wish to use the principle of establishing how many zeros \( \Lambda(\xi; s) \) can have in the cut plane. If we consider \( \Lambda(\xi; s) \) to have \( \kappa(s) \) zeros in the upper half-plane, then clearly, since \( \Lambda(\xi; s) = \Lambda(-\xi; s) \), \( \Lambda(\xi; s) \) will have \( 2\kappa(s) \) zeros in the entire cut plane. To compute \( \kappa(s) \), we need to investigate the change in the argument of \( \Lambda(\xi; s) \) on a contour \( \gamma = \gamma(\theta) \), \( 0 \leq \theta \leq \pi \), and that part of the real axis between \(-R \) and \( R \). Noting that \( \Lambda(\infty; s) = \Gamma(\frac{s}{s+1}) \), we conclude, for \( s \neq 0 \), that in the limit as \( R \to \infty \) there is no change in the argument of \( \Lambda(\xi; s) \) on the semicircle. It therefore follows that \( 2\pi \kappa(s) \) equals the change in the argument of \( \Lambda^+(\xi; s) \), the limiting value of \( \Lambda(\xi; s) \) as \( s \) approaches the real axis from above, and \( t \) proceeds from \(-\infty \) to \(+\infty \). The Plemelj formulas can now be used to deduce the limiting values of \( \Lambda(\xi; s) \). As \( s \) approaches the cut from above \(+\) and below \(-\), we find it convenient to factor \( \Lambda^\pm(\xi; s) \) in the manner,

\[
\Lambda^+(\xi; s) = \int_{1/(s+1)^{3/2}}^{1/(s+1)^{2/2}} \Gamma_1(\xi) \Gamma_2(\xi) \Gamma_3(\xi) \int_{-\infty}^{+\infty} \exp(-\xi z),
\]

(5a)

\[
\Lambda^-(\xi; s) = \int_{1/(s+1)^{3/2}}^{1/(s+1)^{2/2}} \Gamma_1(\xi) \Gamma_2(\xi) \Gamma_3(\xi) \int_{-\infty}^{+\infty} \exp(-\xi z),
\]

(5b)

In Fig. 1, we sketch the three curves \( \Gamma_1 \), \( \Gamma_2 \), and \( \Gamma_3 \) corresponding to \( \Gamma_1(\xi) \), \( \Gamma_2(\xi) \), and \( \Gamma_3(\xi) \) as \( t \) proceeds from \(-\infty \) to \(+\infty \). Note that \( \Gamma_1(0) = \Gamma_2(0) = \Gamma_3(0) = 1 \) and that \( \Gamma_1(\pm \infty) = \Gamma_2(\pm \infty) = \Gamma_3(\pm \infty) = 0 \). If we now divide the \( z \) plane into four open regions such that \( S_0 \) denotes the exterior of \( \Gamma_1 \), \( S_1 \) denotes the finite region bounded by \( \Gamma_1 \) and \( \Gamma_2 \), \( S_2 \) denotes the finite region bounded by \( \Gamma_2 \) and \( \Gamma_3 \), and \( S_3 \) denotes the interior of \( \Gamma_3 \), then it follows that \( \kappa(s) = k \) for \( s \in S_k \), \( k = 0, 1, 2, \) and \( 3 \). Of course, for \( s \) on any of the curves \( \Gamma_1 \), \( \Gamma_2 \), or \( \Gamma_3 \), the boundary values \( \Lambda^\pm(\xi; t) \) vanish.
In the manner of our previous work, we now seek a canonical solution to the Riemann problem defined by the boundary condition

\[ X^+(t; s) = G(t; s) X^-(t; s), \quad t \in (0, \infty), \]

where \( G(t; s) = \exp \left[ \frac{\Lambda^+(t; s)}{\Lambda^-(t; s)} \right] \) is continuous and nonvanishing for \( s \in S_0, S_1, S_2, \) or \( S_3. \) If we write

\[ \log G(t; s) = \log \left| G(t; s) \right| + \theta(t; s) \]

then the analysis of Muskhelishvili allows the canonical solutions to Eq. (6) to be written as

\[ X_k(z; s) = \frac{1}{z_b} \exp \left( \frac{2\pi i}{z_b} \int_0^\infty \log G(t; s) \frac{dt}{t - z} \right), \quad s \in S_k. \]

Now since \( \Lambda(z; s) X_k(z; s) \) is also a solution to Eq. (6), we conclude that \( \Lambda(z; s) X_k(z; s) = \Lambda(z; s) P_k(z; s), \) where \( P_k(z; s) \) is a polynomial in \( z. \)

Since \( X_k(z; s) \) is nonvanishing in the finite \( s \) plane, we can deduce \( P_k(z; s) \) to obtain a factorization of \( \Lambda(z; s) \):

\[ \Lambda(z; s) = \left( \frac{s}{s + 1} \right)^3 X_k(z; s) X_k(-z; s), \quad s \in S_k, \]

(8)

Of course, for \( s \in S_0, \) \( \Lambda(z; s) \) has no zeros, but we can set \( s = 0 \) in Eq. (8) to find explicit results for the zeros of \( \Lambda(z; s) \) for \( s \in S_1: \)

\[ \nu_1(s) = \pm \left( \frac{s + 1}{s} \right)^{1/2} \exp \left[ -\frac{(2\pi i)}{s} \int_0^\infty \left( \log G(t; s) + \frac{2\pi i}{t + 1} \right) \frac{dt}{t} \right], \quad s \in S_1. \]

(9)

For \( s \in S_2, \) we can evaluate Eq. (8) at two points (say \( s = 0 \) and \( i \)) to obtain two equations in \( \nu_1(s) \) and \( \nu_2(s) \) which can be solved to yield

\[ \nu_2(s) = \pm \left( \frac{1}{s} \right)^{1/2} \left[ (A_1 - A_0 - 1 + (-1)^2 \right] \left[ \log (A_0 - 1) \right]^2 + A_1 (A_1 - 2A_0 - 2) \right] \right] \left[ 1^{1/2}, \quad s \in S_2, \]

(10)

where

\[ A_0 = \left( \frac{s + 1}{s} \right)^3 \exp \left[ -\pi i \right] \int_0^\infty \left( \log G(t; s) + \frac{4\pi i}{t + 1} \right) \frac{dt}{t} \]

(11a)

and

\[ A_1 = \left( \frac{s + 1}{s} \right)^3 \Lambda(i; s) \]

\[ \exp \left[ -\pi i \right] \int_0^\infty t \log G(t; s) \frac{dt}{t + 1} \].

(11b)

In a similar manner, for \( s \in S_3, \) we can evaluate Eq. (8) at say, \( s = 0, i, \) and \( 2i \) to obtain three equations that can be solved simultaneously to yield explicit results for \( \nu_1(s), \nu_2(s), \) and \( \nu_3(s). \)

It is clear that to compute the zeros of \( \Lambda(z; s) \) by our technique, one must numerically evaluate the integrals appearing in the solutions, a procedure we believe to be preferable to seeking by iteration the, in general, complex zeros of Eq. (4). We have evaluated all of our explicit solutions, for selected values of \( s, \) using a Gaussian quadrature scheme to represent the required integrals. We found no difficulty in obtaining accuracy to six significant figures, and our results agree with those of Buckner and Feiziger for their \( N = 3 \) model. The two "cutoff" frequencies \( \tilde{\omega}^1 \) and \( \tilde{\omega}^2, \) corresponding to \( \tilde{\omega}^1 = 0 \) and \( \tilde{\omega}^2 = 0, \) as seen from the accompanying figure, were found to be \( \tilde{\omega}^1 = 2.1457 \ldots \) and \( \tilde{\omega}^2 = 0.646453 \ldots, \) which also agree with previously reported values.

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