Discrete spectrum basic to kinetic theory

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The theory of complex variables is used to establish explicit solutions to the transcendental equation that defines the discrete spectrum for a model problem in the kinetic theory of gases.

The time-dependent BGK model in the kinetic theory of gases can be linearized and expressed in the form

$$\left(\frac{\partial}{\partial t}+c_x\frac{\partial}{\partial x}+1\right)h(x,\mathbf{c},t)=\pi^{-3/2}\int h(x,\mathbf{c}',t)$$
$$\times[1+2\mathbf{c}'\cdot\mathbf{c}+\tfrac{2}{3}(c'^2-\tfrac{3}{2})(c^2-\tfrac{3}{2})]\exp(-c'^2)d^3c', \quad (1)$$

where $h(x, \mathbf{c}, t)$ represents the perturbation of the distribution function from the Maxwellian distribution, $\mathbf{c}$, with components c_x, c_y, and c_z and magnitude c, is the velocity, t is the time, and x is the space variable. In the manner of Cercignani,[1] we find that Eq. (1) can be decomposed, by taking appropriate moments, into a set of two coupled integrodifferential equations plus three uncoupled equations. Since the uncoupled equations have been discussed in considerable detail,[1] we wish to report some analysis basic to the set of coupled equations:

$$\left(\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}+1\right)\mathbf{\Psi}(x,\mu,t)=\pi^{-1/2}\int_{-\infty}^{\infty}[\mathbf{Q}(\mu)\tilde{\mathbf{Q}}(\mu')$$
$$+\mathbf{P}(\mu)\tilde{\mathbf{P}}(\mu')]\mathbf{\Psi}(x,\mu',t)\exp(-\mu'^2)\,d\mu'. \quad (2)$$

Here, the elements of the two-vector $\mathbf{\Psi}(x, \mu, t)$ are related[1] to the density and temperature of the gas, and x, μ, and t represent, respectively, the position, velocity component, and time, in dimensionless units. In addition,

$$\mathbf{Q}(\mu)=\begin{bmatrix}(\frac{2}{3})^{1/2}(\mu^2-\frac{1}{2}) & 1\\ (\frac{2}{3})^{1/2} & 0\end{bmatrix},\quad \mathbf{P}(\mu)=(2)^{1/2}\mu\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

Letting

$$\mathbf{\Psi}(x,\mu,t)=\exp(st)\mathbf{\Phi}(\nu,\mu;s)\exp[-(s+1)x/\nu],$$

where s is complex, but $s\neq-1$, and ν is to be determined, we find that the discrete spectrum consists of the zeros $\nu(s)$ of $\Lambda(z;s)=\det\mathbf{\Lambda}(z;s)$, where

$$\mathbf{\Lambda}(z;s)=\mathbf{I}+z\int_{-\infty}^{\infty}\mathbf{\Psi}(\xi;s)\frac{d\xi}{\xi-z}, \quad (3)$$

$$\mathbf{\Psi}(\xi;s)=[1/(s+1)\pi^{1/2}]\tilde{\mathbf{Q}}(\xi)[\mathbf{Q}(\xi)+[2s/(s+1)]\xi^2\mathbf{T}]\exp(-\xi^2),$$

and

$$\mathbf{T}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

It is apparent that the matrix $\mathbf{\Lambda}(z;s)$ is analytic in the z plane cut from $-\infty$ to ∞ along the real axis.

In this note, we develop explicit expressions for the zeros $\nu(s)$ of $\Lambda(z;s)$. The analysis is based on the classical work of Muskhelishvili[2] and makes use of the method we recently reported for solving a class of transcendental equations.[3,4] Upon expanding $\Lambda(z;s)$, we find

$$\Lambda(z;s)=[1/(s+1)^3]\{\tfrac{1}{3}s^2z^2+(s+1)(s-\tfrac{1}{3})(s+\tfrac{1}{2})$$
$$+[\tfrac{2}{3}s^2z^4+\tfrac{1}{3}z^2(4s^2-1)+\tfrac{1}{2}(s+1)(\tfrac{11}{3}s+1)]\Lambda(z)$$
$$+\tfrac{2}{3}(s+1+2sz^2)\Lambda^2(z)\}, \quad (4)$$

where

$$\Lambda(z)=1+\pi^{-1/2}z\int_{-\infty}^{\infty}\exp(-\xi^2)\frac{d\xi}{\xi-z}.$$

We first wish to use the argument principle[5] to establish how many zeros $\Lambda(z;s)$ can have in the cut plane. If we consider $\Lambda(z;s)$ to have $\kappa(s)$ zeros in the upper half-plane, then clearly, since $\Lambda(z;s)=\Lambda(-z;s)$, $\Lambda(z;s)$ will have $2\kappa(s)$ zeros in the entire cut plane. To compute $\kappa(s)$ we need to investigate the change in the argument of $\Lambda(z;s)$ on a contour $z=R\exp(i\theta)$, $0\leq\theta\leq\pi$, and that part of the real axis between $-R$ and R. Noting that $\Lambda(\infty;s)=[s/(s+1)]^3$, we conclude, for $s\neq0$, that in the limit as $R\to\infty$ there is no change in the argument of $\Lambda(z;s)$ on the semicircle. It therefore follows that $2\pi\kappa(s)$ equals the change in the argument of $\Lambda^+(t;s)$, the limiting value of $\Lambda(z;s)$ as z approaches the real axis from above, as t proceeds from $-\infty$ to ∞. The Plemelj formulas can now be used to deduce the limiting values of $\Lambda(z;s)$ as z approaches the cut from above $(+)$ and below $(-)$; we find it convenient to factor $\Lambda^\pm(t;s)$ in the manner,

$$\Lambda^+(t;s)=[1/(s+1)^3][s-\Gamma_1(t)][s-\Gamma_2(t)][s-\Gamma_3(t)] \quad (5a)$$

and

$$\Lambda^-(t;s)=[1/(s+1)^3][s-\overline{\Gamma_1(t)}][s-\overline{\Gamma_2(t)}][s-\overline{\Gamma_3(t)}]. \quad (5b)$$

In Fig. 1, we sketch the three curves Γ_1, Γ_2, and Γ_3 corresponding to $\Gamma_1(t)$, $\Gamma_2(t)$, and $\Gamma_3(t)$ as t proceeds from $-\infty$ to ∞. Note that $\Gamma_1(0)=\Gamma_2(0)=\Gamma_3(0)=-1$ and that $\Gamma_1(\pm\infty)=\Gamma_2(\pm\infty)=\Gamma_3(\pm\infty)=0$. If we now divide the s plane into four open regions such that S_0 denotes the exterior of Γ_1, S_1 denotes the finite region bounded by Γ_1 and Γ_2, S_2 denotes the finite region bounded by Γ_2 and Γ_3, and S_3 denotes the interior of Γ_3, then it follows that $\kappa(s)=k$ for $s\in S_k$, $k=0$, 1, 2, and 3. Of course, for s on any of the curves Γ_1, Γ_2, or Γ_3, the boundary values $\Lambda^\pm(t;s)$ vanish.

In the manner of our previous work[3,4] we now seek a canonical solution to the Riemann problem defined by the boundary condition

$$X^+(t;s)=G(t;s)X^-(t;s),\ t\in(0,\infty), \qquad (6)$$

where $G(t;s)=[\Lambda^+(t;s)]/[\Lambda^-(t;s)]$ is continuous and nonvanishing for $s\in S_0$, S_1, S_2, or S_3. If we write $\log G(t;s)=\ln|G(t;s)|+i\theta(t;s)$ and choose continuous values of the log function such that $\theta(0;s)=-2k\pi$ for $s\in S_k$, $k=0$, 1, 2, and 3, then the analysis of Muskhelishvili[2] allows the canonical solutions to Eq. (6) to be written as

$$X_k(z;s)=\frac{1}{z^k}\exp\left((2\pi i)^{-1}\int_0^\infty \log G(t;s)\,\frac{dt}{t-z}\right),\quad s\in S_k. \qquad (7)$$

Now since $\Lambda(z;s)X_k^{-1}(-z;s)$ is also a solution to Eq. (6), we conclude[2] that $\Lambda(z;s)X_k^{-1}(-z;s)=X_k(z;s)P_k(z;s)$, where $P_k(z;s)$ is a polynomial in z. Since $X_k(z;s)$ is nonvanishing in the finite z plane, we can deduce $P_k(z;s)$ to obtain a factorization of $\Lambda(z;s)$:

$$\Lambda(z;s)=\left(\frac{s}{s+1}\right)^3 X_k(z;s)X_k(-z;s)\prod_{\alpha=1}^{k}[\nu_\alpha^2(s)-z^2],\quad s\in S_k. \qquad (8)$$

Of course, for $s\in S_0$, $\Lambda(z;s)$ has no zeros, but we can set $z=0$ in Eq. (8) to find explicit results for the zeros of $\Lambda(z;s)$ for $s\in S_1$:

$$\nu_1(s)=\pm\left(\frac{s+1}{s}\right)^{3/2}\exp\left[-(2\pi i)^{-1}\times\int_0^\infty\left(\log G(t;s)+\frac{2\pi i}{t+1}\right)\frac{dt}{t}\right],\quad s\in S_1. \qquad (9)$$

For $s\in S_2$, we can evaluate Eq. (8) at two points (say $z=0$ and i) to obtain two equations in $\nu_1^2(s)$ and $\nu_2^2(s)$ which can be solved to yield

$$\nu_\alpha(s)=\pm(\tfrac{1}{2})^{1/2}\{A_1-A_0-1+(-1)^\alpha[(A_0-1)^2+A_1(A_1-2A_0-2)]^{1/2}\}^{1/2},\quad s\in S_2, \qquad (10)$$

where

$$A_0=\left(\frac{s+1}{s}\right)^3\exp\left[-(\pi i)^{-1}\int_0^\infty\left(\log G(t;s)+\frac{4\pi i}{t+1}\right)\frac{dt}{t}\right] \qquad (11a)$$

and

$$A_1=\left(\frac{s+1}{s}\right)^3\Lambda(i;s)\times\exp\left(-(\pi i)^{-1}\int_0^\infty t\log G(t;s)\,\frac{dt}{t^2+1}\right). \qquad (11b)$$

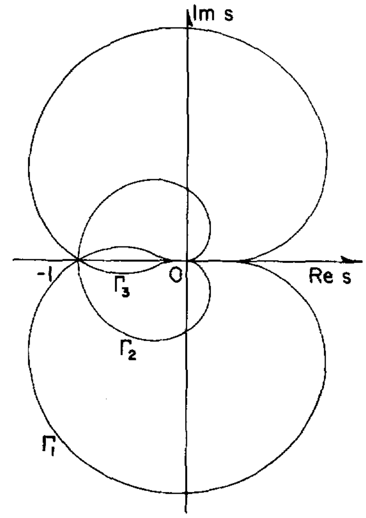

FIG. 1. The s plane.

In a similar manner, for $s\in S_3$, we can evaluate Eq. (8) at, say, $z=0$, i, and $2i$ to obtain three equations that can be solved simultaneously to yield explicit results for $\nu_1(s)$, $\nu_2(s)$, and $\nu_3(s)$.

It is clear that to compute the zeros of $\Lambda(z;s)$ by our technique, one must numerically evaluate the integrals appearing in the solutions, a procedure we believe to be preferable to seeking by iteration the, in general, complex zeros of Eq. (4). We have evaluated all of our explicit solutions, for selected values of s, using a Gaussian quadrature scheme to represent the required integrals. We found no difficulty in obtaining accuracy to six significant figures, and our results agreed with those of Buckner and Ferziger[6] for their $N=3$ model. The two "cutoff" frequencies $\hat{S}_1$ and $\hat{S}_2$, corresponding to $\mathrm{Re}\Gamma_1=0$ and $\mathrm{Re}\Gamma_2=0$, as seen from the accompanying figure, were found to be $\hat{S}_1=2.14517\cdots$ and $\hat{S}_2=0.646453\cdots$, which also agree with previously reported values.[6]

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