Half-space problems in the kinetic theory of gases

C. E. Siewert and J. R. Thomas Jr.

Department of Nuclear Engineering, North Carolina State University, Raleigh, North Carolina 27607

(Received 14 November 1972)

The elementary solutions of the linearized and decomposed BGK equation are used to solve evaporation and heat-conduction problems in a half-space, and values of the microscopic and macroscopic temperature and density slip coefficients are reported, along with the complete temperature and density profiles.

In a recent paper Pao[1] considered two basic problems in the kinetic theory of gases and vapors and reported "exact" values of the microscopic temperature and density slip coefficients. Here, we wish to solve these same two problems, to establish the complete temperature and density profiles, and to give our reasons for believing Pao's "exact" values of the slip coefficients to be in error.

First of all, for the evaporation problem discussed by Pao,[1] we seek a solution of the linearized BGK equation

$$c_x \frac{\partial}{\partial x} f(x, \mathbf{c}) + f(x, \mathbf{c}) = \frac{1}{\pi^{3/2}} [N(x) + (c^2 - \tfrac{3}{2}) T(x) + 2c_x U(x)] \quad (1)$$

subject to the boundary conditions $f(0, \mathbf{c}) = 0$, $c_x > 0$, and

$$\lim_{x\to\infty} \frac{d}{dx} N(x) = 0, \quad \lim_{x\to\infty} \frac{d}{dx} T(x) = 0, \quad (2)$$

where

$$N(x) = \int f(x, \mathbf{c}) \exp(-c^2)\, d^3c, \quad (3a)$$

$$T(x) = \tfrac{2}{3} \int f(x, \mathbf{c}) (c^2 - \tfrac{3}{2}) \exp(-c^2)\, d^3c, \quad (3b)$$

and

$$U(x) = \int f(x, \mathbf{c}) c_x \exp(-c^2)\, d^3c. \quad (3c)$$

As we are concerned only with density and temperature effects, we choose to decompose Eq. (1) in the manner discussed by Cercignani[2] to obtain[3]

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Psi}(x, \mu) = (\pi)^{-1/2} \mathbf{Q}(\mu) \times \int_{-\infty}^{\infty} \tilde{\mathbf{Q}}(\mu') \mathbf{\Psi}(x, \mu') \exp(-\mu'^2)\, d\mu', \quad (4)$$

where $\mathbf{Q}(\mu)$ is a matrix of polynomials.[3]

We note that we now use the variable μ to denote the x component of the velocity and that the density and temperature profiles follow from the 2-vector $\mathbf{\Psi}(x, \mu)$ by way of

$$N(x) = (\pi)^{1/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \int_{-\infty}^{\infty} \mathbf{\Psi}(x, \mu) \exp(-\mu^2)\, d\mu$$

and (5)

$$T(x) = \tfrac{2}{3} (\pi)^{1/2} \int_{-\infty}^{\infty} \begin{bmatrix} \mu^2 - \frac{1}{2} \\ 1 \end{bmatrix}^T \mathbf{\Psi}(x, \mu) \exp(-\mu^2)\, d\mu.$$

We use the superscripts tilde and T interchangeably to denote the transpose operation. To establish the desired results for $N(x)$ and $T(x)$ we thus seek first a solution of Eq. (4) subject to the boundary conditions

$$\mathbf{\Psi}(0, \mu) = -(2U/\pi)\mu \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mu > 0, \quad (6a)$$

$$\lim_{x\to\infty} \frac{d}{dx} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \int_{-\infty}^{\infty} \mathbf{\Psi}(x, \mu) \exp(-\mu^2)\, d\mu = 0, \quad (6b)$$

and

$$\lim_{x\to\infty} \frac{d}{dx} \int_{-\infty}^{\infty} \begin{bmatrix} \mu^2 - \frac{1}{2} \\ 1 \end{bmatrix}^T \mathbf{\Psi}(x, \mu) \exp(-\mu^2)\, d\mu = 0. \quad (6c)$$

Clearly, once $\mathbf{\Psi}(x, \mu)$ is established, the desired density and temperature distributions follow immediately from Eqs. (5) and the various slip coefficients from the definitions[1]

$$\lim_{x\to\infty} N(x) = -2Uc_1, \quad \lim_{x\to\infty} T(x) = -2Ud_1,$$
$$N(0) = -2U\gamma_1, \quad T(0) = -2U\delta_1. \quad (7)$$

The quantities c_1 and d_1 represent the macroscopic density and temperature jumps, whereas γ_1 and δ_1 are the analogous microscopic quantities.

For the heat-transfer problem, also discussed by Pao,[1] we seek a solution of Eq. (1) subject to the boundary condition $f(0, \mathbf{c}) = 0$, $c_x > 0$, and the conditions

$$\lim_{x\to\infty} \frac{d}{dx} N(x) = -1, \quad \lim_{x\to\infty} \frac{d}{dx} T(x) = 1. \quad (8)$$

In addition, for the heat-transfer problem there can be no flow in the x direction, and thus $U = 0$. We therefore seek a solution of Eq. (4) which satisfies the boundary condition

$$\mathbf{\Psi}(0, \mu) = \mathbf{0}, \quad \mu > 0, \quad (9)$$

and the conditions given by Eqs. (8), expressed in terms of $\mathbf{\Psi}(x, \mu)$.

Again the desired slip coefficients follow from the definitions[1]

$$\lim_{x\to\infty} [N(x) + x] = -c_2, \quad \lim_{x\to\infty} [T(x) - x] = d_2,$$
$$N(0) = -\gamma_2, \quad T(0) = \delta_2, \quad (10)$$

where c_2 and d_2 are the macroscopic density and tem-

TABLE I. Temperature and density profiles.

	Evaporation problem		Heat-transfer problem	
x	Temperature	Density	Temperature	Density
0.0	−0.204789	−0.661130	0.853515	−0.396572
0.20	−0.206735	−0.745649	1.20568	−0.724406
0.40	−0.209588	−0.776300	1.47158	−0.975508
0.60	−0.211891	−0.794298	1.71501	−1.20817
0.80	−0.213740	−0.806163	1.94665	−1.43152
1.0	−0.215237	−0.814473	2.17091	−1.64921
1.5	−0.217913	−0.826888	2.71243	−2.17910
2.0	−0.219614	−0.833285	3.23831	−2.69757
2.5	−0.220735	−0.836863	3.75555	−3.20985
3.0	−0.221494	−0.838971	4.26751	−3.71840
3.5	−0.222018	−0.840260	4.77605	−4.22454
4.0	−0.222387	−0.841070	5.28229	−4.72904
4.5	−0.222649	−0.841590	5.78692	−5.23241
6.0	−0.223074	−0.842307	7.29509	−6.73843
20.0	−0.223374	−0.842645	21.3027	−20.7443

perature jumps, and γ_2 and δ_2 are the analogous microscopic jumps.

Kriese *et al.*[3] have expressed a general solution of Eq. (4) as

$$\Psi(x,\mu)=\sum_{\alpha=1}^{2}A_\alpha\Phi_\alpha(\mu)+\sum_{\alpha=3}^{4}A_\alpha\Psi_\alpha(x,\mu)+\sum_{\alpha=1}^{2}\int_{-\infty}^{\infty}A_\alpha(\eta)\Phi_\alpha(\eta,\mu)\exp(-x/\eta)\,d\eta,\quad(11)$$

where A_1, A_2, A_3, A_4, $A_1(\eta)$, and $A_2(\eta)$ are arbitrary coefficients to be determined once appropriate boundary conditions are specified. The vectors required in Eq. (11) have been given explicitly[3] and thus for the sake of space will not be relisted here.

For the evaporation problem we find that $\Psi(x,\mu)$, as given by Eq. (11), will satisfy the conditions given by Eqs. (6b) and (6c) if we take $A_3=A_4=0$, and $A_\alpha(\eta)=0$, $\eta<0$, $\alpha=1$ and 2. Using these results, along with Eq. (11), in Eqs. (5) we find the following expressions for the density and temperature perturbations:

$$N(x)=\pi A_2+(\pi)^{1/2}\int_0^\infty A_2(\eta)\exp(-\eta^2-x/\eta)\,d\eta$$

and (12)

$$T(x)=\pi(\tfrac{2}{3})^{1/2}A_1+(\pi)^{1/2}\int_0^\infty A_1(\eta)\exp(-\eta^2-x/\eta)\,d\eta\,.$$

Clearly then, in order to evaluate the density and temperature profiles and the various slip coefficients, we must determine the expansion coefficients A_1, A_2, $A_1(\eta)$, and $A_2(\eta)$, $\eta>0$. Entering Eq. (11) into Eq. (6a), we find

$$-\frac{2}{\pi}U\mu\begin{bmatrix}1\\0\end{bmatrix}=\sum_{\alpha=1}^{2}\left[A_\alpha\Phi_\alpha(\mu)+\int_0^\infty A_\alpha(\eta)\Phi_\alpha(\eta,\mu)\,d\eta\right],\quad \mu>0,\quad(13)$$

to be the equation from which the expansion coefficients can be determined. Equation (13) is, of course, a system of singular integral equations and can be solved in the manner of Muskhelishvili.[4] We prefer, however, to use the half-range orthogonality theorem of Kriese *et al.*[3] to summarize the results:

$$A_\alpha=-\frac{2}{\pi}U\begin{bmatrix}\delta_{1\alpha}\\\delta_{2\alpha}\end{bmatrix}^T\tilde{\mathbf{H}}_1^{-1}\tilde{\mathbf{H}}_2\begin{bmatrix}0\\1\end{bmatrix},\quad \alpha=1\text{ and }2,\quad(14)$$

and

$$A_\alpha(\eta)=\frac{2U}{\pi}\eta\exp(-\eta^2)\tilde{\mathbf{K}}_\alpha(\eta)\tilde{\mathbf{H}}^{-1}(\eta)\tilde{\mathbf{H}}_1\begin{bmatrix}0\\1\end{bmatrix},\quad \eta>0,\ \alpha=1\text{ and }2.\quad(15)$$

In Eqs. (14) and (15), $\mathbf{H}(\eta)$ is as tabulated,[3]

$$\tilde{\mathbf{H}}_\beta=\pi^{-1/2}\int_0^\infty\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu)\mu^\beta\exp(-\mu^2)\,d\mu,\quad(16)$$

the quantity $\delta_{\alpha\beta}$ represents the Kronecker delta, and the $\mathbf{K}_\alpha(\eta)$ are normalization vectors.[3] With these expansion coefficients determined, the desired results for the density and temperature profiles follow immediately from Eqs. (12), and the various slip coefficients from Eqs. (7) and (12):

$$c_1=\begin{bmatrix}0\\1\end{bmatrix}^T\tilde{\mathbf{H}}_1^{-1}\tilde{\mathbf{H}}_2\begin{bmatrix}0\\1\end{bmatrix},\qquad d_1=(\tfrac{2}{3})^{1/2}\begin{bmatrix}1\\0\end{bmatrix}^T\tilde{\mathbf{H}}_1^{-1}\tilde{\mathbf{H}}_2\begin{bmatrix}0\\1\end{bmatrix},\quad(17)$$

and

$$\gamma_1=\begin{bmatrix}0\\1\end{bmatrix}^T\tilde{\mathbf{H}}_1\begin{bmatrix}0\\1\end{bmatrix},\quad \delta_1=(\tfrac{2}{3})^{1/2}\begin{bmatrix}1\\0\end{bmatrix}^T\tilde{\mathbf{H}}_1\begin{bmatrix}0\\1\end{bmatrix}.\quad(18)$$

Now turning to the heat-transfer problem and following a similar procedure, we find

$$N(x)=\pi A_2-x+(\pi)^{1/2}\int_0^\infty A_2(\eta)\exp(-\eta^2-x/\eta)\,d\eta$$

and (19)

$$T(x)=\pi(\tfrac{2}{3})^{1/2}A_1+x+(\pi)^{1/2}\times\int_0^\infty A_1(\eta)\exp(-\eta^2-x/\eta)\,d\eta,$$

where

$$A_\alpha=\pi^{-1}\begin{bmatrix}\delta_{1\alpha}\\\delta_{2\alpha}\end{bmatrix}^T\tilde{\mathbf{H}}_1^{-1}\tilde{\mathbf{H}}_2\begin{bmatrix}(\frac{3}{2})^{1/2}\\-1\end{bmatrix},\ \alpha=1\text{ and }2,\quad(20)$$

and

$$A_\alpha(\eta)=-\frac{\eta}{\pi}\exp(-\eta^2)\tilde{\mathbf{K}}_\alpha(\eta)\tilde{\mathbf{H}}^{-1}(\eta)\tilde{\mathbf{H}}_1\begin{bmatrix}(\frac{3}{2})^{1/2}\\-1\end{bmatrix},\quad \eta>0,\ \alpha=1\text{ and }2.\quad(21)$$

Similarly, we find the various slip coefficients to be

$$c_2 = -\begin{bmatrix}0\\1\end{bmatrix}^T \tilde{\mathbf{H}}_1^{-1}\tilde{\mathbf{H}}_2 \begin{bmatrix}(\frac{3}{2})^{1/2}\\-1\end{bmatrix},$$

$$d_2 = (\tfrac{2}{3})^{1/2}\begin{bmatrix}1\\0\end{bmatrix}^T \tilde{\mathbf{H}}_1^{-1}\tilde{\mathbf{H}}_2 \begin{bmatrix}(\frac{3}{2})^{1/2}\\-1\end{bmatrix}, \quad (22)$$

and

$$\gamma_2 = -\begin{bmatrix}0\\1\end{bmatrix}^T \tilde{\mathbf{H}}_1 \begin{bmatrix}(\frac{3}{2})^{1/2}\\-1\end{bmatrix}, \delta_2 = (\tfrac{2}{3})^{1/2}\begin{bmatrix}1\\0\end{bmatrix}^T \tilde{\mathbf{H}}_1 \begin{bmatrix}(\frac{3}{2})^{1/2}\\-1\end{bmatrix}. \quad (23)$$

In Table I we list numerical values for the density and temperature profiles for the evaporation problem and the heat-transfer problem, respectively, as computed from Eqs. (12) and (19); $U=\frac{1}{2}$.

In Table II we give, along with Pao's results,[1] our values for the various slip coefficients as computed from Eqs. (17), (18), (22), and (23). A comparison of our results for the microscopic slip coefficients with those of Pao[1] reveals a basic disagreement in the numerical values. The results of Pao[1] are based on the Wiener–Hopf technique, and rely heavily on the theorems of Gohberg and Krein.[5] We believe that there are several typographical errors in Ref. 5 and that these errors may have affected Pao's work.[1]

TABLE II. Values for the slip coefficients.

	Present analysis	Pao [a]
c_1	0.842645	⋯
d_1	0.223375	⋯
γ_1	0.661130	0.666070
δ_1	0.204789	0.193823
c_2	0.744279	⋯
d_2	1.30272	⋯
γ_2	0.396572	0.375336
δ_2	0.853515	0.859893

[a] Ref. 1.

The authors are grateful to Professors E. E. Burniston and T. W. Mullikin for their invaluable discussions regarding this work.

[1]Y. P. Pao, Phys. Fluids **14**, 1340 (1971).
[2]C. Cercignani, *Mathematical Methods in Kinetic Theory* (Plenum, New York, 1969), p. 157.
[3]J. T. Kriese, T. S. Chang, and C. E. Siewert, Int. J. Eng. Sci. (in press).
[4]N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, The Netherlands, 1953), p. 227.
[5]I. C. Gohberg and M. G. Krein, Am. Math. Soc. Transl. **14**, 217 (1960).