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Exact Analytical Solutions of $ze^z = a$

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By means of the theory of complex variables, the solutions of $z \exp z = a$, where a is in general complex, are established analytically, and thereby reduced to elementary quadratures.

I. INTRODUCTION

As discussed, for example, by Wright [1] and Bellman and Cooke [2], the transcendental equation

$$ze^z = a, \qquad a \text{ complex}, \tag{1}$$

is basic to the analysis of a class of differential-difference equations and, more recently [3], has been found essential to certain studies in the theory of population growth. As an application of our reported procedure [4] for solving a class of transcendental equations, we wish to develop here the analysis required to reduce the solutions of equation (1) to elementary quadratures.

II. ANALYSIS

It is immediately apparent that the solutions z_k of equation (1) are given by the zeros of the functions

$$\Lambda_k(z) = \alpha - z - \log z + 2k\pi i, \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

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where

$$\alpha = \log a, \qquad a \neq 0, \tag{3}$$

and, here, $\log z$ denotes the principal branch of the log-function. We note therefore that $\Lambda_k(z)$ is analytic in the complex plane cut along the negative real axis. In addition, the limiting values of $\Lambda_k(z)$ as z approaches the negative real axis from above (+) and below (−) can be computed at once from equation (2):

$$\Lambda_k^+(t) = \alpha - t - \ln|t| + (2k-1)\pi i, \qquad t \in (-\infty, 0), \tag{4a}$$

and

$$\Lambda_k^-(t) = \alpha - t - \ln|t| + (2k+1)\pi i, \qquad t \in (-\infty, 0). \tag{4b}$$

It is clear from equations (3) and (4) that neither $\Lambda_k^+(t)$ nor $\Lambda_k^-(t)$ can vanish on the cut $t \in (-\infty, 0)$ except for the two special cases

(i) $a \in (-1/e, 0)$ and $k = 0$

and

(ii) $a \in (-1/e, 0)$ and $k = -1$.

Since, for the special cases (i) and (ii), equations (4) yield

$$\Lambda_0^+(t) = \Lambda_{-1}^-(t), \qquad a \in \left(-\frac{1}{e}, 0\right), \tag{5}$$

the desired solutions are the roots of

$$\ln|a| = x + \ln|x|, \qquad a \in \left(-\frac{1}{e}, 0\right), \qquad x \in (-\infty, 0). \tag{6}$$

Elementary considerations are sufficient to show that equation (6) has only two solutions x_0 and x_{-1} and further that $x_{-1} < -1 < x_0 < 0$. Of course, for $a = -e^{-1}$ the two solutions of equation (6) coalesce at -1; whereas for $a = 0$, the solutions $x_0 = 0$ and $x_{-1} = -\infty$ can be obtained by a limiting process.

We shall discuss the special cases (i) and (ii) separately and first consider all values of a and k such that

$$\{a, k\} \in D \Rightarrow k = 0, \pm 1, \pm 2, \pm 3, \ldots, \qquad \text{if} \qquad a \notin \left[-\frac{1}{e}, 0\right]$$

or

$$\{a, k\} \in D \Rightarrow k = 1, \pm 2, \pm 3, \pm 4, \ldots, \qquad \text{if} \qquad a \in \left(-\frac{1}{e}, 0\right).$$

The argument principle [5] can now be used to show, for $\{a, k\} \in D$, that $\Lambda_k(z)$ has precisely one zero, say z_k, in the cut plane. If therefore follows that

$$F_k(z) = \frac{\Lambda_k(z)}{z_k - z}, \qquad \{a, k\} \in D, \tag{7}$$

is analytic and nonvanishing in the same cut plane. Now since

$$F_k(\infty) = 1, \qquad \{a, k\} \in D, \tag{8}$$

we conclude that $F_k(z)$ is a canonical solution of the Riemann problem [6, 7]

$$F_k^+(t) = G_k(t) F_k^-(t), \qquad t \in (-\infty, 0), \quad \{a, k\} \in D, \tag{9}$$

where

$$G_k(t) = \frac{\Lambda_k^+(t)}{\Lambda_k^-(t)}, \qquad \{a, k\} \in D, \tag{10}$$

is the Riemann coefficient. Equation (9) is, naturally, an immediate consequence of equation (7).

If we now define $G_k(-\infty) = G(0) = 1$, then it is apparent that $G_k(t)$ is continuous on the negative real axis, but fails to be Hölder continuous at $t = 0$. It therefore follows from the work of Simonenko [7] that the canonical solution

$$F_k(z) = \exp\left[-\frac{1}{2\pi i}\int_0^\infty \log G_k(-t)\frac{dt}{t+z}\right], \qquad \{a, k\} \in D, \tag{11}$$

will satisfy equation (9) pointwise. Note that $\log G_k(-t)$ is continuous $t \in (0, \infty)$, and such that $\log G_k(-\infty) = \log G_k(0) = 0$.

With $F_k(z)$ given by equation (11), we can now solve equation (7) immediately to obtain the explicit closed-form result

$$z_k = z + (\log a - z - \log z + 2k\pi i)\exp\left[\frac{1}{2\pi i}\int_0^\infty \log G_k(-t)\frac{dt}{t+z}\right], \qquad \{a, k\} \in D. \tag{12}$$

We note that our solutions given by equation (12) contain a free parameter z which can be assigned any convenient value in the complex plane. The choice of z in equation (12) can alter the computational merit of the resulting expression; however, we can set $z = 1$ in equation (12) to obtain the concise solutions

$$z_k = 1 + (\log a - 1 + 2k\pi i)\exp\left[\frac{1}{2\pi i}\int_0^\infty \log G_k(-t)\frac{dt}{t+1}\right], \qquad \{a, k\} \in D. \tag{13}$$

Of course, the improper integrals appearing in equations (12) and (13) can be avoided by introducing the integration variable $\tau = t(1 + t)^{-1}$ to obtain

$$z_k = z + (\log a - z - \log z + 2k\pi i) \times \exp\left[\frac{1}{2\pi i}\int_0^1 \log G_k\left(\frac{\tau}{\tau - 1}\right)\frac{d\tau}{\tau(1-\tau) + z(1-\tau)^2}\right], \tag{14}$$

or

$$z_k = 1 + (\log a - 1 + 2k\pi i)\exp\left[\frac{1}{2\pi i}\int_0^1 \log G_k\left(\frac{\tau}{\tau-1}\right)\frac{d\tau}{1-\tau}\right], \quad \{a, k\} \in D. \tag{15}$$

In the event that $a \in (-\infty, -1/e)$, and thus $\alpha = \ln|a| + \pi i$, we note that since

$$\log G_{-1-k}(-t) = -\overline{\log G_k(-t)}, \qquad a \in \left(-\infty, -\frac{1}{e}\right), \quad k = 0, \pm 1, \pm 2, \pm 3,\ldots, \tag{16}$$

it follows at once from equation (12) that

$$z_{-1-k} = \overline{z_k}, \qquad a \in \left(-\infty, -\frac{1}{e}\right), \qquad k = 0, 1, 2, 3,\ldots, \tag{17}$$

and thus the desired solutions occur in conjugate pairs z_k and $\overline{z_k}$, $k = 0, 1, 2, 3,\ldots$. In a similar manner, we also find that

$$z_{-1-k} = \overline{z_k}, \qquad a \in \left(-\frac{1}{e}, 0\right), \qquad k = 1, 2, 3,\ldots, \tag{18}$$

so that the solutions corresponding to $\{a, k\} \in D$ are z_k and $\overline{z_k}$, $k = 1, 2, 3,\ldots$. Having now resolved all of the cases for which $\{a, k\} \in D$, we wish to consider the two special cases (i) and (ii) corresponding to $a \in (-1/e, 0)$, $k = 0$ and $k = 1$. As previously mentioned, for $a \in (-1/e, 0)$, the solutions deriving from all other values of k are given by z_k and $\overline{z_k}$, $k = 1, 2, 3,\ldots$. Since the final two solutions are the roots of equation (6), we consider the function

$$\Omega(z) = \ln|a| + \pi i - z - \log z, \qquad a \in \left(-\frac{1}{e}, 0\right), \tag{19}$$

where by $\log z$ we now denote that branch of the log-function in the plane cut along the *positive* real axis, such that $0 < \arg z < 2\pi$. With the log-function so defined, it follows that $\Omega(z)$ is analytic in the plane cut along the *positive* real axis and that the limiting values take the forms

$$\Omega^+(t) = \ln|a| - t - \ln t + \pi i, \qquad t \in (0, \infty), \qquad a \in \left(-\frac{1}{e}, 0\right), \tag{20a}$$

and

$$\Omega^-(t) = \ln|a| - t - \ln t - \pi i, \qquad t \in (0, \infty), \qquad a \in \left(-\frac{1}{e}, 0\right). \tag{20b}$$

The argument principle [5] can now be used to show that $\Omega(z)$ has exactly two zeros, say x_0 and x_{-1}, in the cut plane, and thus

$$E(z) = -\frac{\Omega(z)}{(x_0 - z)(x_{-1} - z)}, \qquad a \in \left(-\frac{1}{e}, 0\right), \tag{21}$$

is analytic and nonvanishing in the finite cut plane. We conclude, therefore, that $E(z)$ is a canonical solution of the Riemann problem

$$E^+(t) = G(t)\,E^-(t), \qquad t \in (0, \infty), \qquad a \in \left(-\frac{1}{e}, 0\right), \tag{22}$$

where the Riemann coefficient is

$$G(t) = \frac{\Omega^+(t)}{\Omega^-(t)} = \exp[2i \arg \Omega^+(t)]. \tag{23}$$

We can now solve the Riemann problem defined by equations (22) and (23) to obtain the (appropriately normalized) canonical solution

$$E(z) = \frac{1}{z} \exp\left[\frac{1}{\pi}\int_0^\infty [\arg \Omega^+(t) - \pi]\frac{dt}{t - z}\right], \qquad a \in \left(-\frac{1}{e}, 0\right), \tag{24}$$

which can be entered into equation (21) to yield

$$(x_0 - z)(x_{-1} - z) = -z\Omega(z) \exp\left[-\frac{1}{\pi}\int_0^\infty [\arg \Omega^+(t) - \pi]\frac{dt}{t - z}\right]. \tag{25}$$

If we evaluate equation (25) at two convenient points, say $z = \beta$ and $z = \gamma$, then the two resulting equations can be solved simultaneously to yield

$$x_0 = -B(\beta, \gamma) + \sqrt{B^2(\beta, \gamma) - C(\beta, \gamma)}, \qquad a \in \left(-\frac{1}{e}, 0\right) \tag{26a}$$

and

$$x_{-1} = -B(\beta, \gamma) - \sqrt{B^2(\beta, \gamma) - C(\beta, \gamma)}, \qquad a \in \left(-\frac{1}{e}, 0\right), \tag{26b}$$

where

$$B(\beta, \gamma) = \frac{1}{2}\left[\frac{K(\beta) - K(\gamma) - \beta^2 + \gamma^2}{\beta - \gamma}\right] \tag{27a}$$

and

$$C(\beta, \gamma) = \left[\frac{\beta K(\gamma) - \gamma K(\beta) + \beta\gamma(\beta - \gamma)}{\beta - \gamma}\right]. \tag{27b}$$

In addition, we have introduced

$$K(z) = -z\Omega(z) \exp\left[-\frac{1}{\pi}\int_0^\infty [\arg \Omega^+(t) - \pi]\frac{dt}{t - z}\right], \tag{28}$$

which, after changing the integration variable to $\tau = t(1 + t)^{-1}$, we can write as

$$K(z) = -z[\ln|a| + \pi i - z - \log z] \times \exp\left[-\frac{1}{\pi}\int_0^1 \left[\arg \Omega^+\left(\frac{\tau}{1-\tau}\right) - \pi\right]\frac{d\tau}{\tau(1-\tau) - z(1-\tau)^2}\right]. \tag{29}$$

The solutions given by equations (26) clearly contain two free parameters β and γ, which can perhaps be used to computational advantage; if we take $\beta = -1$ and $\gamma = -2$, then

$$K(-1) = (\ln|a| + 1) \exp\left[-\frac{1}{\pi}\int_0^1 \left[\arg \Omega^+\left(\frac{\tau}{1-\tau}\right) - \pi\right]\frac{d\tau}{1-\tau}\right] \tag{30a}$$

and

$$K(-2) = 2(\ln|a| + 2 - \ln 2) \times \exp\left[-\frac{1}{\pi}\int_0^1 \left[\arg \Omega^+\left(\frac{\tau}{1-\tau}\right) - \pi\right]\frac{d\tau}{(1-\tau)(2-\tau)}\right] \tag{30b}$$

can be used with

$$B(-1, -2) = \tfrac{1}{2}[K(-1) - K(-2) + 3] \tag{31a}$$

and

$$C(-1, -2) = 2K(-1) - K(-2) + 2 \tag{31b}$$

to yield the explicit solutions

$$x_0 = -\tfrac{1}{2}[K(-1) - K(-2) + 3] + \tfrac{1}{2}[[K(-1) - K(-2)]^2 + 1 - 2[K(-1) + K(-2)]]^{1/2}, \quad a \in \left(-\frac{1}{e}, 0\right), \tag{32a}$$

and

$$x_{-1} = -\tfrac{1}{2}[K(-1) - K(-2) + 3] - \tfrac{1}{2}[[K(-1) - K(-2)]^2 + 1 - 2[K(-1) + K(-2)]]^{1/2}, \quad a \in \left(-\frac{1}{e}, 0\right). \tag{32b}$$

Some elementary considerations can now be used to show that x_0 and x_{-1} as given by equations (32), are both real and negative.

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