Journal of Applied Mathematics and Physics (ZAMP)
Vol. 24, 1973 Birkhäuser Verlag Basel

On Double Zeros of $x - \tanh(ax + b)$

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I. Introduction

In a recent paper, Siewert and Essig [1] reported closed-form solutions of the transcendental equation

$$x = \tanh(ax + b), \tag{1}$$

for nonzero real values of a and b, that is of interest in the molecular field theory of ferromagnetism. It can be concluded, say from elementary graphical considerations, that equation (1) can, for a given value of b, have a real repeated solution only for a particular value of $a > 0$. Here we wish to report an exact analytical expression for that value of a.

If equation (1) is to have a repeated solution, then clearly equation (1) and

$$a = \frac{1}{1 - x^2} \tag{2}$$

must be solved simultaneously. We note therefore that the analogous problem of finding b, for a given value of a, such that equation (1) has a repeated solution, does not require the solution to a transcendental equation. However, for the considered problem we can enter equation (2) into equation (1) to obtain

$$x = (x^2 - 1)(b - \tanh^{-1} x), \tag{3}$$

and thus we wish to consider b to be given, solve equation (3) analytically, and subsequently enter that result into equation (2) to obtain the desired result.

II. Analysis

If we now let $x\xi = 1$, we can write equation (2) as

$$a_\alpha = \frac{\xi_\alpha^2}{\xi_\alpha^2 - 1}, \tag{4}$$

where ξ_α is a zero of the sectionally analytic function

$$\Omega(\xi) = \xi + (\xi^2 - 1)\left[b + \frac{1}{2}\int_{-1}^{1} \frac{d\mu}{\mu - \xi}\right]. \tag{5}$$

Note that we have used the principal branch of arctanh z to define $\Omega(\xi)$. It is now a straightforward matter to show that the limiting values $\Omega^{\pm}(t)$ of $\Omega(\xi)$ as ξ approaches the cut $[-1, 1]$ from above $(+)$ and below $(-)$ cannot vanish. Further, it is clear that $\Omega(\xi)$ has three zeros, ξ_0, ξ_1 and ξ_2, in the cut plane, that $\Omega(\xi)$ cannot have a purely imaginary zero, that $\Omega(\xi)$ has only one real zero, say ξ_0, and that ξ_1 and ξ_2 are a complex-conjugate pair. We conclude therefore that equation (1) can have a real repeated solution only for

$$a=a_0=\frac{\xi_0^2}{\xi_0^2-1}, \tag{6}$$

and thus we now wish to develop an explicit expression for ξ_0.

Since we shall require the limiting values of $\Omega(\xi)$ as ξ approaches the cut, we can use the Plemelj formulae [2] to obtain, from equation (5),

$$\Omega^{\pm}(t)=t+(t^2-1)[b-\tanh^{-1} t \pm \tfrac{1}{2}\pi i], \qquad t \in (-1, 1). \tag{7}$$

It thus follows that

$$F(\xi)=\frac{\Omega(\xi)}{(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)} \tag{8}$$

is a canonical solution of the Riemann problem [2] defined by

$$\Phi^+(t)=G(t)\,\Phi^-(t), \qquad t \in (-1, 1), \tag{9}$$

where

$$G(t)=\frac{\Omega^+(t)}{\Omega^-(t)}=e^{2\,i \arg \Omega^+(t)}. \tag{10}$$

In other words, $F(\xi)$ is analytic in the cut plane, is nonvanishing in the finite plane, and satisfies equation (9). If we now let

$$\Theta(t)=\arg \Omega^+(t)=\tan^{-1}\left[\frac{\pi(t^2-1)}{2[t+(t^2-1)(b-\tanh^{-1} t)]}\right], \tag{11}$$

with $\Theta(-1)=-\pi$, then a canonical solution of equation (9) can be written as [2]

$$\Phi(\xi)=\frac{1}{\xi+1}\exp\left[\frac{1}{\pi}\int_{-1}^{1}\Theta(t)\,\frac{dt}{t-\xi}\right]. \tag{12}$$

Now since canonical solutions can differ only by a multiplicative constant, we conclude, from considerations at infinity, that

$$F(\xi)=b\,\Phi(\xi), \tag{13}$$

and thus we can enter equation (13) into equation (8) to obtain

$$(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)=P(\xi), \tag{14}$$

where $P(\xi)$ is a known function:

$$P(\xi) = \frac{1}{b}\,\Omega(\xi)(\xi+1)\exp\left[-\frac{1}{\pi}\int_{-1}^{1}\Theta(t)\frac{dt}{t-\xi}\right]. \tag{15}$$

Equation (14) is, of course, an identity in the ξ plane, and thus we can evaluate that equation at three convenient points, say $\xi=\alpha$, β and γ, and solve the resulting three equations simultaneously to obtain ξ_0, ξ_1, and ξ_2. We list here the explicit results which follow from taking $\xi=\alpha t$, $\xi=\beta t$, and $\xi=\gamma t$ and subsequently observing the limit as $t\to\infty$:

$$a = \frac{\xi_0^2}{\xi_0^2-1}, \tag{16}$$

where

$$\xi_0 = S_1 + S_2 - \tfrac{1}{3}A_2, \tag{17}$$

$$S_j = [R - (-1)^j[R^2+Q^3]^{\frac{1}{2}}]^{\frac{1}{3}}, \tag{18}$$

$$R = \tfrac{1}{6}[A_1A_2 - 3A_0] - \tfrac{1}{27}A_2^3, \tag{19}$$

and

$$Q = \tfrac{1}{3}A_1 - \tfrac{1}{9}A_2^2. \tag{20}$$

In addition

$$A_0 = \frac{2-3b}{3b} + J_1 + \frac{1}{2}J_0^2 + J_2 - J_0 + J_0J_1 + \frac{1}{6}J_0^3, \tag{21}$$

$$A_1 = -1 + J_0 + J_1 + \tfrac{1}{2}J_0^2 \tag{22}$$

and

$$A_2 = 1 + J_0, \tag{23}$$

where

$$J_\alpha = \frac{1}{\pi}\int_{-1}^{1}\Theta(t)\,t^\alpha\,dt. \tag{24}$$

We have used a Gaussian integration procedure to evaluate our explicit solutions, and accuracy to eight significant figures was achieved quite straightforwardly.

Acknowledgment

The authors are grateful to a reviewer for suggesting this problem.

References

[1] C. E. Siewert and C. J. Essig, Z. angew. Math. Phys. *24*, 281 (1973).
[2] N. I. Muskhelishvili, *Singular Integral Equations* (Nordhoff, Groningen, The Netherlands) 1953.

Abstract

The theory of complex variables is used to develop an exact closed-form expression, in terms of elementary quadratures, for that value of a, for a given value of b, for which the function $F(x)=x-\tanh(ax+b)$ has a real double zero.

Zusammenfassung

Mit Hilfe der Theorie der komplexen Veränderlichen wurde ein exakter geschlossener Ausdruck gefunden, der durch Quadraturen den Wert von a gibt, für einen gegebenen Wert von b, für welche die Funktion

$$F(x)=x-\tanh(ax+b)$$

eine reelle doppelte Nullstelle besitzt.

(Received: November 3, 1972)