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AN EXACT SOLUTION OF A TRANSCENDENTAL EQUATION BASIC TO THE THEORY OF INTERMEDIATE RESONANCE ABSORPTION OF NEUTRONS

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Abstract—The theory of complex variables is used to solve explicitly a transcendental equation essential to the intermediate-resonance–absorption theory for neutrons. The closed-form solutions are expressed ultimately in terms of elementary quadratures.

I. INTRODUCTION

IN A classic paper on the theory of neutron resonance absorption, GOLDSTEIN and COHEN (1962), introduced the λ-method for analysing absorption intermediate to the usual narrow and wide resonance approximations. GOLDSTEIN and COHEN (1962) used a successive approximations method and a variational approach to develop two techniques for computing intermediate resonance integrals; they reported, however, that the solutions, though concise, were difficult to use because the solutions to certain transcendental equations were required in the final results.

Here we shall make use of the method recently reported by SIEWERT and BURNISTON (1972, 1973) to solve explicitly the transcendental equation basic to GOLDSTEIN and COHEN'S (1962) successive approximations method for computing intermediate resonance integrals.

To establish the notation, we start with the basic equations developed by GOLDSTEIN and COHEN (1962); we first seek a solution β_λ to the transcendental equation

$$\lambda = 1 - \frac{\arctan x_{1\lambda}}{x_{1\lambda}}, \tag{1}$$

where

$$x_{1\lambda} = \frac{2E_r(1-\alpha)}{\Gamma(\beta_1 + \beta_\lambda)}, \tag{2}$$

$$\beta_1 = \sqrt{1 + \frac{\sigma_0}{s + \sigma_p}}, \tag{3}$$

and

$$\lambda = \left[\sigma_0 \frac{\Gamma_\gamma}{\Gamma} + s(1 - \beta_\lambda^2)\right]\left[\sigma_p(\beta_\lambda^2 - 1) - \sigma_0 \frac{\Gamma_n}{\Gamma}\right]^{-1}, \tag{4}$$

and once β_λ is established, the desired λ follows immediately from equation (4).

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Since the characteristic parameters, E_r, α, Γ, Γ_γ, Γ_n, σ_0, s and σ_p, have been defined by GOLDSTEIN and COHEN (1962), further comments about these parameters are not required here; we note, however, that equation (1) is applicable only for the principal branch of the arctangent function.

If we introduce

$$\sigma = \frac{\Gamma}{2E_r(1-\alpha)} \tag{5a}$$

and

$$\omega = \frac{\sigma_p}{s+\sigma_p} \tag{5b}$$

and let

$$z_\alpha = i\sigma(\beta_1 + \beta_\lambda), \tag{6}$$

then equation (1) can be expressed as

$$z_\alpha - iU - \omega(z_\alpha^2 - iUz_\alpha + V)\tanh^{-1}\frac{1}{z_\alpha} = 0, \tag{7}$$

where

$$U = 2\sigma\beta_1 \tag{8a}$$

and

$$V = \frac{\sigma_0}{\sigma_p}\sigma^2\left[\frac{\Gamma_n}{\Gamma} - \frac{\sigma_p}{s+\sigma_p}\right]. \tag{8b}$$

We consider equation (7) the transcendental equation to be solved (for z_α) and thus can use equations (4) and (6) to obtain the required λ:

$$\lambda = \frac{\dfrac{\Gamma_\gamma}{\Gamma} - \dfrac{s}{\sigma_p+s} - \dfrac{s}{\sigma_0}\left[\dfrac{2\beta_1 z_\alpha}{\sigma}i - \dfrac{z_\alpha^2}{\sigma^2}\right]}{\dfrac{\sigma_p}{\sigma_p+s} - \dfrac{\Gamma_n}{\Gamma} + \dfrac{\sigma_p}{\sigma_0}\left[\dfrac{2\beta_1 z_\alpha}{\sigma}i - \dfrac{z_\alpha^2}{\sigma^2}\right]}. \tag{9}$$

II. THE NUMBER OF ZEROS OF $\Lambda(z)$ IN THE CUT PLANE

We now wish to solve equation (7) explicitly. If we let

$$\Lambda(z) = P_1(z) - P_2(z)\tfrac{1}{2}\int_{-1}^{1}\frac{d\mu}{\mu - z}, \tag{10}$$

where

$$P_1(z) = z - iU \tag{11a}$$

and

$$P_2(z) = -\omega(z^2 - iUz + V), \tag{11b}$$

we conclude that $\Lambda(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis and that

$$\Lambda(z) \to (1-\omega)z, \quad \text{as} \quad |z| \to \infty.$$

We note also that all zeros of $\Lambda(z)$ in the cut plane are solutions of equation (7). Our first task therefore is to determine the number (κ) of zeros of $\Lambda(z)$ in the cut plane. We shall make use of the argument principle, (AHLFORS, 1953) and thus we must compute the change in the argument of $\Lambda(z)$ as a contour enclosing the cut plane is traversed. We consider $\omega < 1$, since $\omega = 1$ is a simple special case, and therefore conclude that

$$\Delta_C \arg \Lambda(z) = 2\pi, \tag{12}$$

where $\Delta_C \arg \Lambda(z)$ denotes the change in the argument of $\Lambda(z)$ about a circle centered at the origin and with unbounded radius C. To complete this computation, we must consider the change in the argument of $\Lambda(z)$ as a contour just enclosing the branch cut $[-1, 1]$ is traversed.

If in equation (10) we allow z to approach the branch cut from above (+) and below (−), the Plemelj formulae (MUSKHELISHVILI, 1953) yield the boundary values of $\Lambda(z)$:

$$\Lambda^{\pm}(t) = P_1(t) + P_2(t)[\operatorname{arctanh} t \mp \tfrac{1}{2}\pi i]. \tag{13}$$

Now write

$$\Lambda^{+}(t) = R(t)e^{i\theta(t)}, \qquad t \in [-1, 1], \tag{14a}$$

and

$$\Lambda^{-}(t) = L(t)e^{i\phi(t)}, \qquad t \in [-1, 1], \tag{14b}$$

where, from equations (11) and (13),

$$\tan \theta(t) = \frac{A(t)}{B(t)} \tag{15a}$$

and

$$\tan \phi(t) = \frac{O(t)}{E(t)}, \tag{15b}$$

with

$$A(t) = -U + \omega\left[Ut \operatorname{arctanh} t + \frac{\pi}{2}(V + t^2)\right], \tag{16a}$$

$$B(t) = t - \omega\left[(V + t^2) \operatorname{arctanh} t - \frac{\pi}{2} Ut\right], \tag{16b}$$

$$O(t) = -U + \omega\left[Ut \operatorname{arctanh} t - \frac{\pi}{2}(V + t^2)\right], \tag{16c}$$

and

$$E(t) = t - \omega\left[(V + t^2) \operatorname{arctanh} t + \frac{\pi}{2} Ut\right]. \tag{16d}$$

It follows now, since there is no change in $\arg \Lambda(z)$ about arbitrarily small arcs about the endpoints −1 and 1, that

$$\kappa = \frac{1}{2\pi}[2\pi + \langle\theta(t)\rangle - \langle\phi(t)\rangle], \tag{17}$$

where $\langle f(t)\rangle$ is used to denote the change in $f(t)$ as t proceeds from −1 to 1.

In order to compute κ from equation (17) some rather careful analysis of the functions $A(t)$, $B(t)$, $O(t)$ and $E(t)$, for $t \in [-1, 1]$, is required, since the results depend, naturally, rather intricately on the parameters ω, U and V defined by equations (5b) and (8). Here we consider real values of the parameters such that $0 < \omega < 1$, $U > 0$, and $|V| > 0$, and in Tables 1 and 2 we summarize our conclusions regarding the number κ of zeros (in the cut plane) of $\Lambda(z)$. We believe, however, that several additional comments are required.

Table 1.—The number of zeros κ of $\Lambda(z)$ in cut plane for $\omega \in (0, 1)$, $U > 0$ and $V > 0$

$\frac{\omega\pi}{2} V - U$	$B(t_0)$	κ
>0	$[A(t_0) \neq 0]$	2
<0	>0	3
<0	<0	1

Table 2.—The number of zeros κ of $\Lambda(z)$ in cut plane for $\omega \in (0,1)$, $U > 0$ and $V < 0$

$\frac{\omega\pi}{2} \lvert V \rvert - U$	$E(t_1)$	$E(t_2)$	κ
>0	>0	>0	2
>0	>0	<0	4
>0	$\gtrless 0$	$[0(t_2) \neq 0]$	2
>0	$[0(t) \neq 0]$	$[0(t) \neq 0]$	2
>0	<0	<0	2
<0	>0	$[0(t_2) \neq 0]$	1
<0	<0	$[0(t_2) \neq 0]$	3

First of all, for $V > 0$ and $A(0) < 0$, we find that κ depends on $B(t_0)$ where $t_0 \in (0, 1)$ is defined by $A(t_0) = 0$, *i.e.*, $\kappa = 1$ for $B(t_0) < 0$, and $\kappa = 3$ for $B(t_0) > 0$. For $V < 0$, we find that κ depends on $E(t_1)$ and $E(t_2)$, t_1 and $t_2 \in (0, 1)$ with $t_2 > t_1$, where $O(t_1) = O(t_2) = 0$. Of course, $A(t) = 0$, $t \in (0, 1)$ and $O(t) = 0$, $t \in (0, 1)$, are in fact transcendental equations, which could be solved if necessary; however, to determine κ it is sufficient to establish only the algebraic signs of $B(t_0)$, for $V > 0$, or $E(t_1)$ and $E(t_2)$, for $V < 0$, and this can, in general, be accomplished either by employing an analytical bounding procedure or numerically, if preferred. We note also that our Tables 1 and 2 do not include several (unlikely) special cases which require rather careful deductions to be resolved.

III. EXPLICIT RESULTS

Since the method (Siewert and Burniston, 1972; Burniston and Siewert, 1973) we use here to solve the considered transcendental equation is based on seeking zeros in the cut plane of an even function $F(z)$, we now introduce

$$F(z) = \Lambda(z)\Lambda(-z) \tag{18}$$

and note, obviously, that if $\Lambda(z)$ has κ zeros in the cut plane then $F(z)$ has 2κ zeros in the cut plane. In the manner of our previous work, we now seek a canonical solution to the Riemann problem (Muskhelishvili, 1953) defined by the boundary condition

$$X^+(t) = G(t)X^-(t), \qquad t \in (0, 1), \tag{19}$$

where

$$G(t) = \frac{F^+(t)}{F^-(t)}, \tag{20}$$

with $G(0) = G(1) = 1$, is continuous and nonvanishing (for the cases listed in Tables 1 and 2).

We note that $G(t)$ has unit modulus, and thus if we write

$$G(t) = \exp\,[2i\,\oplus(t)] \tag{21}$$

and choose continuous values of $\oplus(t)$ such that $\oplus(0) = 0$, for all κ, then the analysis of MUSKHELISHVILI (1953) allows us to write canonical solutions to the Riemann problem defined by equation (19) as

$$X_1(z) = \exp\left[\frac{1}{\pi}\int_0^1 \oplus(t)\frac{\mathrm{d}t}{t-z}\right], \qquad \kappa = 1, \tag{22a}$$

$$X_2(z) = \left(\frac{1}{1-z}\right)\exp\left[\frac{1}{\pi}\int_0^1 \oplus(t)\frac{\mathrm{d}t}{t-z}\right], \qquad \kappa = 2, \tag{22b}$$

or, in general,

$$X_\kappa(z) = \left(\frac{1}{1-z}\right)^{(\kappa-1)}\exp\left[\frac{1}{\pi}\int_0^1 \oplus(t)\frac{\mathrm{d}t}{t-z}\right], \qquad \kappa = 1, 2, 3 \text{ or } 4. \tag{22c}$$

If we now observe that $F(z)X^{-1}(-z)$ is also a solution to equation (19), then we can write (MUSKHELISHVILI, 1953)

$$F(z)X^{-1}(-z) = X(z)\mathscr{P}(z) \tag{23}$$

where $\mathscr{P}(z)$ is a polynomial in z. Since $X(z)$ is nonvanishing in the finite plane, we deduce that

$$\mathscr{P}_1(z) = (1-\omega)^2[\nu_1{}^2 - z^2], \qquad \kappa = 1, \tag{24a}$$

$$\mathscr{P}_2(z) = (1-\omega)^2[\nu_1{}^2 - z^2][\nu_2{}^2 - z^2], \qquad \kappa = 2, \tag{24b}$$

or, in general with $\pm\nu_\alpha$ denoting the zeros of $F(z)$,

$$\mathscr{P}_\kappa(z) = (1-\omega)^2\prod_{\alpha=1}^{\kappa}(\nu_\alpha{}^2 - z^2), \qquad \kappa = 1, 2, 3 \text{ or } 4. \tag{24c}$$

This completes the factorization of $F(z)$:

$$F(z) = (1-\omega)^2 X_\kappa(z)X_\kappa(-z)\prod_{\alpha=1}^{\kappa}(\nu_\alpha{}^2 - z^2), \qquad \kappa = 1, 2, 3 \text{ or } 4. \tag{25}$$

For $\kappa = 1$, we can set $z = 0$ [since $G(t)$ is Hölder continuous at the origin] in equation (25) to obtain the explicit result

$$\nu_1 = \pm\left[\frac{\pi^2}{4}V^2\omega^2 - U^2\right]^{1/2}(1-\omega)^{-1}\exp\left[-\frac{1}{\pi}\int_0^1 \oplus(t)\frac{\mathrm{d}t}{t}\right], \qquad \kappa = 1, \tag{26}$$

where, in general,

$$\oplus(t) = \tan^{-1}\left[\frac{A(t)E(t) - B(t)O(t)}{B(t)E(t) + A(t)O(t)}\right]. \tag{27}$$

For the case $\kappa = 2$, equation (25) can be evaluated at two points, say $z = 0$ and $z = i$, to yield two equations in ν_1^2 and ν_2^2 which can be solved to give the results

$$\nu_1 = \pm \frac{1}{\sqrt{2}} [A(i) - A(0) - 1 + \{[A(0) - A(i)]^2 + 1 - 2[A(i) + A(0)]\}^{1/2}]^{1/2}, \qquad \kappa = 2, \quad (28a)$$

and

$$\nu_2 = \pm \frac{1}{\sqrt{2}} [A(i) - A(0) - 1 - \{[A(0) - A(i)]^2 + 1 - 2[A(i) + A(0)]\}^{1/2}]^{1/2}, \qquad \kappa = 2, \quad (28b)$$

where

$$A(0) = (1 - \omega)^{-2}\left[\omega^2 V^2 \frac{\pi^2}{4} - U^2\right] \exp\left[-\frac{2}{\pi}\int_0^1 \Theta(t)\frac{dt}{t}\right] \quad (29a)$$

and

$$A(i) = 2(1 - \omega)^{-2} F(i) \exp\left[-\frac{2}{\pi}\int_0^1 t\,\Theta(t)\frac{dt}{t^2 + 1}\right]. \quad (29b)$$

In a similar manner, we can evaluate equation (25) for $\kappa = 3$ at, say, $z = 0$, $z = i$ and $z = 2i$, and by successive eliminations obtain the bi-cubic equation

$$\nu_\alpha^6 - \frac{\nu_\alpha^4}{12}[A(2i) - 8A(i) + 3A(0) - 60] - \frac{\nu_\alpha^2}{12}[A(2i) - 32A(i) + 15A(0) - 48] - A(0) = 0, \qquad \kappa = 3, \quad (30)$$

where we now require

$$A(2i) = 25(1 - \omega)^{-2} F(2i) \exp\left[-\frac{2}{\pi}\int_0^1 t\,\Theta(t)\frac{dt}{t^2 + 4}\right]. \quad (31)$$

Equation (30) can now be solved analytically by using the standard technique (ABRAMOWITZ and STEGUN, 1964) for solving cubic equations to obtain the three results ν_1^2, ν_2^2 and ν_3^2, which in turn yield immediately the six zeros of $F(z)$ for $\kappa = 3$. Finally, for $\kappa = 4$, we can evaluate equation (25) at, say, $z = 0$, i, $2i$ and $3i$ to obtain, after elimination, a quartic equation in ν_α^2 which again can be solved analytically (ABRAMOWITZ and STEGUN, 1964) to yield ν_1^2, ν_2^2, ν_3^2 and ν_4^2 and subsequently the eight zeros of $F(z)$ for $\kappa = 4$.

We have succeeded in finding analytical solutions for all of the zeros $\{z_\alpha\}_{2\kappa}$ of $F(z)$ in the plane cut from -1 to 1 along the real axis. It therefore follows, since $F(z) = \Lambda(z)\Lambda(-z)$, that half of this set $\{z_\alpha\}_\kappa$ corresponds, in fact, to closed-form results, by means of equation (9), for λ. Some rather elementary analysis can now be used to deduce which elements of the set $\{z_\alpha\}_{2\kappa}$ are zeros of $\Lambda(z)$ and which are zeros of $\Lambda(-z)$.

IV. EXAMPLE CALCULATION

In order to illustrate the application of the preceding solutions, we consider, as did GOLDSTEIN and COHEN (1962), the 192 eV resonance of ^{238}U. We have converted GOLDSTEIN and COHEN's (1962) data to our notation. We find that for this case V is positive, and since we can argue that $B(t_0)$ is positive, we conclude from Table 1 that $\kappa = 3$.

We have used an improved Gaussian quadrature scheme (KRONROD, 1965) to evaluate numerically the integrals required in equations (29) and (31) and subsequently have solved equation (30) analytically to obtain

$$\begin{aligned} \nu_1 &= \pm i0{\cdot}8553096 \\ \nu_2 &= \pm 1{\cdot}010503 \pm i0{\cdot}01155536 \\ \nu_3 &= \pm 1{\cdot}010503 \mp i0{\cdot}01155536 \end{aligned} \tag{32}$$

Some elementary considerations now reveal that of the six zeros of $F(z)$ given by equation (32), the subset

$$\begin{aligned} z_1 &= + i0{\cdot}8553096 \\ z_2 &= 1{\cdot}010503 + i0{\cdot}01155536 \\ z_3 &= -1{\cdot}010503 + i0{\cdot}01155536 \end{aligned} \tag{33}$$

are our computed zeros of $\Lambda(z)$, and thus from equation (9) we find the *three* solutions to equation (1):

$$\begin{aligned} \lambda_1 &= 0{\cdot}261673 \\ \lambda_2 &= -1{\cdot}45924 + i0{\cdot}389937 \\ \lambda_3 &= -1{\cdot}45924 - i0{\cdot}389937 \end{aligned} \tag{34}$$

We note that GOLDSTEIN and COHEN (1962) reported the value $\lambda = 0{\cdot}264$ and that we have shown by iteration that each of the solutions given by equation (34) satisfies equation (1) to within the accuracy reported. Of course, for GOLDSTEIN and COHEN's (1962) use of equation (1), the real λ's are of principal interest.

We have also evaluated our analytical solutions of equation (1) cast into the form of equation (7) to confirm, for an example of each of the cases in Tables 1 and 2, results obtained by solving equation (7) iteratively.

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