# Heat transfer between parallel plates with arbitrary surface accommodation

J. R. Thomas Jr.*
*Department of Nuclear Engineering, North Carolina State University, Raleigh, North Carolina 27607*

T. S. Chang†
*Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

C. E. Siewert†
*Institut für Struktur der Materie, Universität Karlsruhe, Karlsruhe, West Germany*
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Elementary solutions of the coupled pair of integrodifferential equations arising from the decomposition of the linearized BGK equation in the kinetic theory of gases are used to solve the problem of heat transfer between parallel plates with arbitrary surface accommodation. A coupled pair of Fredholm equations is derived, and rapidly convergent iterative solutions are constructed. These solutions are then used to obtain accurate values of the heat flux between the plates and the temperature and density profiles, for various values of the accommodation coefficient and inverse Knudsen number. Numerical results for the heat flux are presented and compared to existing variational solutions. Also, explicit results for the temperature and density profiles are given.

## I. INTRODUCTION

The problem of heat transfer in a rarefied gas has attracted considerable attention over the years.[1–20] Quite recently, Bassanini *et al.*[9] applied a variational technique to the linearized Bhatnagar–Gross–Krook (BGK) equation to calculate the heat flux between parallel plates for purely diffuse reflection and for arbitrary surface accommodation.[10] For the purely diffuse case, the variational results appeared to be quite accurate by comparison to a direct numerical solution of the integral form of the transport equation.[9] Based on this success, similar accuracy was inferred, without actual comparison to another computational technique, for the more general case.[10]

Recently, Kriese *et al.*[11] reported a general solution of the vector transport equation describing temperature–density effects which arise after the linearized BGK equation is decomposed in the manner discussed by Cercignani.[12] With this analysis available it now becomes feasible to establish a truly accurate solution to the parallel-plate heat-transfer problem.

In the present paper, the general analysis of Kriese *et al.* is used to reduce the solution of the heat-transfer problem to the solution of a coupled pair of Fredholm equations. We have found that iterative solutions to these equations converge very rapidly and thus have computed the solution to benchmark accuracy. We compare the results for the heat flux to the variational results of Bassanini *et al.*[9,10]

Unlike the variational technique, the present method also yields accurate temperature and density profiles between the plates; these results are also presented in tabular forms.

## II. ELEMENTARY SOLUTIONS

In order to establish the required notation and formalism, we first wish to review and summarize briefly the principal results of Kriese *et al.*[11]

As discussed by Cercignani,[12] the linearized BGK equation can be decomposed such that when only temperature and density effects are of interest, an investigation of a pair of coupled integrodifferential equations is sufficient to yield the desired results. On the other hand, should additional information about the gas be of interest, the three remaining uncoupled equations in Cercignani's decompositional scheme must be investigated. Since here we are concerned with temperature–density effects, we base our analysis on the pair of coupled equations, written here in matrix notation[11] as,

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x,\mu) + \mathbf{\Psi}(x,\mu) = \pi^{-1/2}\mathbf{Q}(\mu) \times \int_{-\infty}^{\infty} \tilde{\mathbf{Q}}(\mu')\, \mathbf{\Psi}(x,\mu') \exp(-\mu'^2)\, d\mu', \quad (1)$$

where

$$\mathbf{Q}(\mu) = \begin{bmatrix} (\frac{2}{3})^{1/2}(\mu^2 - \frac{1}{2}) & 1 \\ (\frac{2}{3})^{1/2} & 0 \end{bmatrix}, \quad (2)$$

$x$ is the distance from the origin measured in units of the mean-free path, $\mu$ is the nondimensional $x$ component of velocity, and the components of

$$\mathbf{\Psi}(x,\mu) = \begin{bmatrix} \Psi_1(x,\mu) \\ \Psi_2(x,\mu) \end{bmatrix} \quad (3)$$

are simply related to the dimensionless perturbations in the number density and temperature of the gas:

$$\Delta\rho(x) = \pi^{-1} \int_{-\infty}^{\infty} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \boldsymbol{\Psi}(x, \mu) \exp(-\mu^2)\, d\mu, \quad (4)$$

and

$$\Delta T(x) = \frac{2}{3\pi} \int_{-\infty}^{\infty} \begin{bmatrix} \mu^2 - \frac{1}{2} \\ 1 \end{bmatrix}^T \boldsymbol{\Psi}(x, \mu) \exp(-\mu^2)\, d\mu. \quad (5)$$

In addition, the heat flux $q$ is given by

$$q = \frac{Q}{Q_{\mathrm{fm}}} = \left(\frac{2-\alpha}{\alpha}\right) \frac{1}{2(\pi)^{1/2}} \times \int_{-\infty}^{\infty} \begin{bmatrix} \mu^2 + 1 \\ 1 \end{bmatrix}^T \boldsymbol{\Psi}(x, \mu)\, \mu \exp(-\mu^2)\, d\mu, \quad (6)$$

where $Q_{\mathrm{fm}}$ represents the heat flux appropriate to free-molecular conditions. In this work we use the superscripts $T$ and tilde interchangeably to denote the transpose operation. Also, in Eq. (6) $\alpha$ is the accommodation coefficient, or the fraction of the molecules striking the wall which are diffusely reflected.

In order to solve the considered boundary-value problem, based on Eq. (1), we first require the elementary solutions of Eq. (1) that were developed by Kriese *et al.*[11] We thus write a general solution to Eq. (1) as

$$\boldsymbol{\Psi}(x, \mu) = \sum_{\alpha=1}^{2} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=3}^{4} A_\alpha \boldsymbol{\Psi}_\alpha(x, \mu) + \sum_{\alpha=1}^{2} \int_{-\infty}^{\infty} A_\alpha(\eta)\, \boldsymbol{\Phi}_\alpha(\eta, \mu) \exp(-x/\eta)\, d\eta, \quad (7)$$

where the basis vectors $\boldsymbol{\Phi}_\alpha(\mu)$, $\alpha = 1, 2$, $\boldsymbol{\Psi}_\alpha(x, \mu)$, $\alpha = 3, 4$, and $\boldsymbol{\Phi}_\alpha(\eta, \mu)$, $\alpha = 1, 2$, are given explicitly in Ref. 11. We note that completeness and orthogonality theorems basic to the elementary solutions of Eq. (1) are also given in Ref. 11.

## III. BOUNDARY CONDITIONS

For confined gases, boundary conditions appropriate to describe the interaction of gas molecules and solid surfaces have been discussed by a number of authors (see, e.g., the recent reviews of Cercignani[13] and Williams[14]). For the case of a mixture of diffuse and specular reflection, it is conventional to write the reflected distribution from the wall as

$$f(\mathbf{r}_s, \mathbf{v}, t) = (1-\alpha)\, f(\mathbf{r}_s, \mathbf{v}^*, t) + \alpha B' f_w(\mathbf{v}), \quad (8)$$

for $\mathbf{v}\cdot\mathbf{n} > 0$. Here $f(\mathbf{r}, \mathbf{v}, t)$ is the particle distribution at position $\mathbf{r}$, velocity $\mathbf{v}$, and time $t$, $f_w(\mathbf{v})$ is a Maxwellian characterized by the temperature $T_w$ and density $n_w$ of the wall, $\mathbf{v}^*$ is the reflection of $\mathbf{v}$ in the tangent plane at $\mathbf{r}_s$, $\mathbf{n}$ is the unit normal at $\mathbf{r}_s$ pointing into the gas, and $\alpha$ is the accommodation coefficient. If the condition of particle conservation is to be satisfied at the wall, the constant $B'$ in Eq. (16) must satisfy the constraint

$$\int \mathbf{v}\cdot\mathbf{n}\, f(\mathbf{r}_s, \mathbf{v}, t)\, d\mathbf{v} = 0. \quad (9)$$

We wish to consider the problem of heat transfer between parallel plates spaced a distance $d$ apart. We take the origin of the coordinate system midway between the two plates, which then lie at $x = \pm\delta/2$, with

$$\delta = d/[\theta(2RT_0)^{1/2}]. \quad (10)$$

Here, $\theta$ is the mean-free time, $T_0$ is the mean temperature, and $R$ is the gas constant. The temperatures of the two plates are $T_1 = T_0 + \Delta T$ (at $x = -\delta/2$) and $T_2 = T_0 - \Delta T$ (at $x = \delta/2$).

Linearizing the boundary condition (8) about a Maxwellian based on the mean temperature $T_0$, and decomposing the resulting expression in the same manner used to derive Eq. (1), we find that the boundary conditions at the two plates can be written as

$$\boldsymbol{\Psi}(\mp\tfrac{1}{2}\delta, \pm\mu) = (1-\alpha)\, \boldsymbol{\Psi}(\mp\tfrac{1}{2}\delta, \mp\mu) \pm \alpha\pi^{1/2} \times \begin{bmatrix} \mu^2 + B \\ 1 \end{bmatrix}, \quad \mu \in (0, \infty), \quad (11)$$

where $B$ is to be determined by mass conservation.

At this point we might note that the boundary condition given by Eq. (11) is not the same as that used in Ref. 10. Bassanini *et al.*[10] have approximated the outgoing distribution at the boundary by a local Maxwellian with the density and temperature determined by mass and energy conservation, a form convenient for use with the variational technique. Our boundary condition is, of course, the more classical one for this problem, and is quite easily handled with our method of analysis.

## IV. BASIC ANALYSIS

In view of Eq. (11), we find it convenient to impose on $\boldsymbol{\Psi}(x, \mu)$ the antisymmetry condition

$$\boldsymbol{\Psi}(x, \mu) = -\boldsymbol{\Psi}(-x, -\mu). \quad (12)$$

Applying the condition given by Eq. (12) to the general solution given by Eq. (7) allows us to write the general solution for the present problem as

$$\boldsymbol{\Psi}(x, \mu) = \sum_{\alpha=3}^{4} A_\alpha \boldsymbol{\Psi}_\alpha(x, \mu) + \sum_{\alpha=1}^{2} \int_0^{\infty} A_\alpha(\eta) [\boldsymbol{\Phi}_\alpha(\eta, \mu) e^{-x/\eta} - \boldsymbol{\Phi}_\alpha(\eta, -\mu) e^{x/\eta}] d\eta. \quad (13)$$

Because of the form of $\Psi(x, \mu)$ as given by Eq. (13), it is clear that we need only consider Eq. (11) for, say $x = -\delta/2$. Further, we note that there must be no net flow in the $x$ direction, and thus we can write

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \int_{-\infty}^{\infty} \Psi(x, \mu)\mu \exp(-\mu^2)d\mu = 0. \quad (14)$$

If we now enter Eq. (13) into Eq. (14), we find the following relation between two of the desired expansion coefficients:

$$A_4 = -(\tfrac{2}{3})^{1/2}A_3. \quad (15)$$

To determine the remaining expansion coefficients we can enter Eqs. (13) and (15) into Eq. (11), for $x = -\delta/2$, to obtain

$$A_3[\mathbf{\Phi}_1(\mu) - (\tfrac{2}{3})^{1/2}\mathbf{\Phi}_2(\mu)]\left[(2-\alpha)\mu + \frac{\alpha\delta}{2}\right] + \sum_{\beta=1}^{2}\int_0^{\infty} A_\beta(\eta)[\mathbf{\Phi}_\beta(\eta, \mu)g(\delta, \eta) - \mathbf{\Phi}_\beta(\eta, -\mu) \times g(-\delta, \eta)]d\eta = p_1\mathbf{\Phi}_1(\mu) + p_2\mathbf{\Phi}_2(\mu), \mu > 0, \quad (16)$$

where

$$g(\delta, \eta) = \exp(\delta/2\eta) + (1-\alpha)\exp(-\delta/2\eta), \quad (17)$$

$$p_1 = \alpha(3\pi/2)^{1/2}, \quad (18a)$$

and

$$p_2 = \alpha\pi^{1/2}(B + \tfrac{1}{2}). \quad (18b)$$

We note that Eq. (16) is, in fact, a system of singular integral equations that must be solved to yield the expansion coefficients; however, before proceeding to regularize Eq. (16) to establish a system of nonsingular equations for $A_3$, $A_1(\eta)$, and $A_2(\eta)$, we observe that the final results follow immediately from Eq. (13). Thus, if we enter Eq. (13) into Eq. (6), we find that the heat flux $q$ is given by

$$q = \tfrac{5}{8}(\tfrac{2}{3})^{1/2}[(2-\alpha)/\alpha]A_3. \quad (19)$$

In a similar manner, we can enter Eq. (13) into Eqs. (4) and (5) to obtain

$$\Delta T(x) = -\left(\frac{2}{3\pi}\right)^{1/2} A_3 x - \frac{2}{\pi}\int_0^{\infty} A_1(\eta) \times \sinh(x/\eta)\exp(-\eta^2)\, d\eta \quad (20)$$

and

$$\Delta\rho(x) = \left(\frac{2}{3\pi}\right)^{1/2} A_3 x - \frac{2}{\pi}\int_0^{\infty} A_2(\eta) \times \sinh(x/\eta)\exp(-\eta^2)\, d\eta. \quad (21)$$

As discussed in Ref. 11, the half-range orthogonality theorem and the constructed adjoint vectors can be used to regularize singular integral equations of the form of Eq. (16). The use of these adjoint vectors and the orthogonality theorem,[11] which is, of course, equivalent to the methods of Muskhelishvili,[21] leads directly to the regular integral equations that are basic to our iterative numerical solution. Thus, we find

$$A_3 = \frac{p_1}{N_3} + N_3^{-1}\int_0^{\infty} \frac{\mu\exp(-\mu^2)}{\pi^{1/2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \tilde{\mathbf{H}}_1^{-1}\mathbf{H}^{-1}(\mu) \times \begin{bmatrix} (\frac{3}{2})^{1/2}A_1(\mu) \\ A_2(\mu) \end{bmatrix} g(-\delta, \mu)\, d\mu, \quad (22)$$

where

$$N_3 = (2-\alpha)\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \tilde{\mathbf{H}}_1^{-1}\tilde{\mathbf{H}}_2 \begin{bmatrix} 1 \\ -(\frac{2}{3})^{1/2} \end{bmatrix} + \frac{\alpha\delta}{2}. \quad (23)$$

Here, $\mathbf{H}(\mu)$ is the $\mathbf{H}$ matrix defined and tabulated in Ref. 11 and

$$\tilde{\mathbf{H}}_\alpha = \pi^{-1/2}\int_0^{\infty} \tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu)\exp(-\mu^2)\mu^\alpha d\mu. \quad (24)$$

In a similar manner we obtain the two regular equations

$$A_\beta(\eta) = (2-\alpha)A_3\eta\exp(-\eta^2)\tilde{\mathbf{K}}_\beta(\eta)\tilde{\mathbf{H}}^{-1}(\eta)\tilde{\mathbf{H}}_1 \times \begin{bmatrix} 1 \\ -(\frac{2}{3})^{1/2} \end{bmatrix}[g(\delta, \eta)]^{-1} + \frac{\eta e^{-\eta^2}}{\pi^{1/2}}\tilde{\mathbf{K}}_\beta(\eta) \times \frac{\tilde{\mathbf{H}}^{-1}(\eta)}{g(\delta, \eta)}\int_0^{\infty} \mathbf{H}^{-1}(\mu)\begin{bmatrix} (\frac{3}{2})^{1/2}A_1(\mu) \\ A_2(\mu) \end{bmatrix}\mu\exp(-\mu^2) \times g(-\delta, \mu)\frac{d\mu}{\eta + \mu}, \quad \eta > 0, \quad \beta = 1, 2. \quad (25)$$

Here,

$$\mathbf{K}_1(\eta) = [N(\eta)]^{-1}\begin{bmatrix} (\frac{3}{2})^{1/2}N_{22}(\eta) \\ -N_{12}(\eta) \end{bmatrix} \text{ and}$$

$$\mathbf{K}_2(\eta) = [N(\eta)]^{-1}\begin{bmatrix} -(\frac{3}{2})^{1/2}N_{12}(\eta) \\ N_{11}(\eta) \end{bmatrix}, \quad (26)$$

where the $N_{ij}(\eta)$ and $N(\eta)$ are normalization integrals given explicitly by Kriese *et al.*[11]

We observe that Eq. (22) expresses $A_3$ in terms of $A_1(\eta)$ and $A_2(\eta)$ and thus that equation can be entered into Eqs. (25) to yield two regular integral equations dependent only on $A_1(\eta)$ and $A_2(\eta)$. It should be noted that, since $\mathbf{\Phi}_2(\mu)$ is orthogonal to each of the three adjoint vectors used, the constant $B$ does not appear in the equations to be solved numerically, Eqs. (22) and (25), and consequently need not be determined explicitly to evaluate Eqs. (19), (20), and (21).

Although Eqs. (22) and (25) appear prohibitively difficult to solve analytically, we have found that they can be solved by iteration quite straightforwardly to yield the desired numerical results.

## V. COMPUTATIONAL RESULTS

As was shown in the previous section, the desired results for the heat flux and the temperature and density profiles are dependent upon the expansion coefficients $A_3$, $A_1(\eta)$, and $A_2(\eta)$, $\eta \in (0, \infty)$, and thus we now wish to discuss the manner in which these expansion coefficients can be computed from Eqs. (22) and (25). Of course, approximate semianalytical solutions of Eqs. (22) and (25) can be developed, for example by neglecting the integral terms in Eq. (25); however, here we wish to report the results of applying an iterative numerical technique to solve Eqs. (22) and (25).

TABLE I. Heat flux between parallel plates for $\alpha = 1.0$.

| $\delta$ | Present theory | Reference 9 | |
|---|---|---|---|
| | | Numerical | Variational |
| 0.0 | 1.000000 | ... | ... |
| 0.01 | 0.992484 | 0.9925 | 0.9925 |
| 0.1 | 0.935159 | 0.9352 | 0.9352 |
| 0.5 | 0.768262 | 0.7682 | 0.7683 |
| 1.0 | 0.640853 | 0.6405 | 0.6409 |
| 1.25 | 0.593761 | 0.5933 | 0.5939 |
| 1.5 | 0.553805 | 0.5532 | 0.5539 |
| 1.75 | 0.519333 | 0.5194 | 0.5194 |
| 2.0 | 0.489203 | 0.4893 | 0.4894 |
| 2.5 | 0.438866 | 0.4390 | 0.4391 |
| 3.0 | 0.398324 | 0.3985 | 0.3986 |
| 4.0 | 0.336724 | 0.3370 | 0.3370 |
| 5.0 | 0.291918 | 0.2923 | 0.2922 |
| 7.0 | 0.230810 | 0.2314 | 0.2311 |
| 10.0 | 0.175788 | 0.1767 | 0.1760 |

First of all, we have used an elementary transformation to map the interval $(0, \infty)$ onto $(0, 1)$ and subsequently used a Gaussian quadrature integration procedure to evaluate all required integrals. Of course, the **H** matrix is required before the iterative solution of Eqs. (22) and (25) can be developed, and hence we have used the method reported in Ref. 11 to establish the **H** matrix at the required nodal points. The calculations were performed in double-precision arithmetic on an IBM 370/165 computer, and having established the **H** matrix, we terminated the iteration procedure when successively calculated values of the coefficients $A_1(\eta)$ and $A_2(\eta)$ differed by less than one part in $10^{12}$.

To attempt to establish confidence in our calculations a number of additional calculations were performed. For example, we have taken the moments of Eqs. (16) using the calculated values of $A_3$, $A_1(\eta)$, and $A_2(\eta)$ and these moment checks were consistent with the accuracy of our reported results. We have not attempted to verify Eq. (16) pointwise since such a verification would require the numerical evaluation of principal-value integrals—a procedure we prefer to avoid. We note that Eqs. (22) and (25) do not obviously reduce to trivial forms for $\delta = 0$ (free molecular conditions), and thus we have computed the heat flux $q$, for the case $\delta = 0$, and agreement with the correct value of unity was achieved to 12 significant figures. Finally, the number of nodal points in the quadrature scheme was repeatedly doubled until such calculations failed to alter the reported values of the heat flux and the temperature and density profiles.

In Table I we list computed values of the heat flux

TABLE II. Heat flux between parallel plates.

| $\delta$ | $\alpha = 0.826$ | | $\alpha = 0.759$ | |
|---|---|---|---|---|
| | Present theory | Reference 10 | Present theory | Reference 10 |
| 0.01 | 0.994675 | 0.9947 | 0.995360 | 0.9954 |
| 0.10 | 0.952722 | 0.9535 | 0.958414 | 0.9593 |
| 0.50 | 0.821214 | 0.8250 | 0.839639 | 0.8443 |
| 1.0 | 0.712079 | 0.7173 | 0.738150 | 0.7448 |
| 1.25 | 0.669706 | 0.6751 | 0.698040 | 0.7051 |
| 1.5 | 0.632830 | 0.6383 | 0.662804 | 0.6700 |
| 1.75 | 0.600300 | 0.6058 | 0.631460 | 0.6387 |
| 2.0 | 0.571299 | 0.5767 | 0.603306 | 0.6104 |
| 2.5 | 0.521603 | 0.5266 | 0.554588 | 0.5614 |
| 3.0 | 0.480375 | 0.4850 | 0.513700 | 0.5200 |
| 4.0 | 0.415508 | 0.4194 | 0.448465 | 0.4539 |
| 5.0 | 0.366501 | 0.3698 | 0.398409 | 0.4030 |
| 7.0 | 0.296937 | 0.2993 | 0.326153 | 0.3295 |
| 10.0 | 0.231353 | 0.2329 | 0.256665 | 0.2588 |

TABLE III. Numerical values of the temperature and density perturbations for $\delta = 0.10$.

| $x/\delta$ | $\alpha = 1.0$ | | $\alpha = 0.759$ | |
|---|---|---|---|---|
| | Temperature | Density | Temperature | Density |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.05 | −0.012558 | 0.011294 | −0.0084399 | 0.0076026 |
| 0.10 | −0.025160 | 0.022629 | −0.016913 | 0.015237 |
| 0.15 | −0.037850 | 0.034049 | −0.025456 | 0.022938 |
| 0.20 | −0.050683 | 0.045604 | −0.034109 | 0.030745 |
| 0.25 | −0.063721 | 0.057355 | −0.042923 | 0.038706 |
| 0.30 | −0.077050 | 0.069385 | −0.051964 | 0.046885 |
| 0.35 | −0.090794 | 0.081813 | −0.061330 | 0.055378 |
| 0.40 | −0.10516 | 0.094839 | −0.071185 | 0.064342 |
| 0.45 | −0.12058 | 0.10888 | −0.081871 | 0.074113 |
| 0.50 | −0.13909 | 0.12594 | −0.095021 | 0.086295 |

TABLE IV. Numerical values of the temperature and density perturbations for $\delta=7.0$.

| $x/\delta$ | $\alpha=1.0$ | | $\alpha=0.759$ | |
|---|---|---|---|---|
| | Temperature | Density | Temperature | Density |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.05 | −0.075060 | 0.074462 | −0.065334 | 0.064686 |
| 0.10 | −0.15025 | 0.14901 | −0.13080 | 0.12947 |
| 0.15 | −0.22570 | 0.22375 | −0.19657 | 0.19445 |
| 0.20 | −0.30159 | 0.29880 | −0.26281 | 0.25978 |
| 0.25 | −0.37818 | 0.37435 | −0.32982 | 0.32565 |
| 0.30 | −0.45585 | 0.45067 | −0.39800 | 0.39237 |
| 0.35 | −0.53523 | 0.52825 | −0.46806 | 0.46048 |
| 0.40 | −0.61761 | 0.60809 | −0.54142 | 0.53105 |
| 0.45 | −0.70618 | 0.69277 | −0.62160 | 0.60700 |
| 0.50 | −0.82192 | 0.80097 | −0.73213 | 0.70929 |

for $\alpha = 1$ (complete accommodation) and selected values of $\delta$; the numerical and variational results of Bassanini *et al.*[9] are also given in Table I. We observe that the variational results are in closer agreement with our results.

In Table II we report similar results for $\alpha = 0.826$ and $\alpha = 0.759$, and compare our solutions to the variational results of Ref. 10. It is important to keep in mind that the results compared in Table II were computed, not only by different analytical techniques, but also using different boundary conditions. For $\alpha = 1$ (Table I) the boundary conditions reduce to identical forms. It might be noted that in Tables I and II, the fractional difference between our results and those of Bassanini *et al.*[9] is less than 2% for all reported cases.

Finally, we report in Tables III and IV our computed values of the temperature and density profiles for four typical cases, as computed from Eqs. (20) and (21).

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*Present address: Babcock and Wilcox Co., Lynchburg, Virginia 24505.
†Permanent address: Departments of Nuclear Engineering and Engineering Mechanics, North Carolina State University, Raleigh, North Carolina 27607.

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