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ELEMENTARY SOLUTIONS OF COUPLED MODEL EQUATIONS IN THE KINETIC THEORY OF GASES

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Abstract—The method of elementary solutions is employed to solve two coupled integrodifferential equations sufficient for determining temperature–density effects in a linearized BGK model in the kinetic theory of gases.

Full-range completeness and orthogonality theorems are proved for the developed normal modes and the infinite-medium Green's function is constructed as an illustration of the full-range formalism.

The appropriate homogeneous matrix Riemann problem is discussed, and half-range completeness and orthogonality theorems are proved for a certain subset of the normal modes. The required existence and uniqueness theorems relevant to the **H** matrix, basic to the half-range analysis, are proved, and an accurate and efficient computational method is discussed. The half-space temperature-slip problem is solved analytically, and a highly accurate value of the temperature-slip coefficient is reported.

1. INTRODUCTION

THERE EXISTS in the kinetic theory of gases a class of one-dimensional problems for which the transverse momentum and heat-transfer effects can be separated by projecting the basic kinetic equation describing the particle distribution function onto certain properly chosen directions in a Hilbert space. The resulting expression describing the heat-transfer and compressibility effects is a vector integrodifferential equation with a matrix kernel similar in form to one studied previously by Bond and Siewert[4] and Burniston and Siewert[5] in connection with the scattering of polarized light. It can be shown that such a vector integrodifferential equation admits a general solution similar to that suggested by Case[7] for scalar transport problems and applied by Cercignani[12] to kinetic equations.

We develop in this paper the elementary solutions to the vector integrodifferential equation basic to a linearized, constant collision frequency (BGK) model suggested by Bhatnagar *et al.*[3] and Welander[29]. The elementary solutions, some of which are generalized functions[14], can be shown to possess rather general full-range and half-range completeness and orthogonality properties. The expansion (or completeness) theorems are proved by reducing a system of singular integral equations to an equivalent matrix Riemann problem and subsequently making use of the theory of Mandžavidze and Hvedelidze[20] and Muskhelishvili[21] to establish the solubility of the resulting equations.

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As an application of our established analysis, we construct in this paper the infinite-medium Green's function useful for developing particular solutions to the basic transport equation. We also make use of the half-range expansion theorem to solve the notoriously difficult temperature slip problem considered previously [2, 19, 23, 28, 29] by approximate methods. Our solution permits an accurate computation of the 'temperature slip coefficient' which may be used to evaluate the merits of approximate techniques.

2. THE KINETIC MODEL AND LINEARIZATION

Basically, the BGK model is constructed by replacing the collision integral in the Boltzmann equation by a more tractable relaxation term; we therefore write

$$\left[\frac{\partial}{\partial\tau}+\mathbf{u}\cdot\nabla\right]f(\mathbf{y},\mathbf{u},\tau)=\nu[\hat{f}(\mathbf{y},\mathbf{u},\tau)-f(\mathbf{y},\mathbf{u},\tau)], \tag{2.1}$$

where $f(\mathbf{y},\mathbf{u},\tau)$ is the particle distribution function, $\mathbf{y}$ is the position vector, $\mathbf{u}$ is the particle velocity, τ is the time, and ν is a characteristic collision frequency. To ensure that the model conserves particles, momentum and energy, we require that

$$\int[\hat{f}(\mathbf{y},\mathbf{u},\tau)-f(\mathbf{y},\mathbf{u},\tau)]\mathbf{U}\,\mathrm{d}^3u=\mathbf{0}, \tag{2.2}$$

where the integration is to be taken over all velocity space and $\mathbf{U}$ is a five-element vector with components 1, u_1, u_2, u_3, and u^2, the collisional invariants. Here u_α, $\alpha=1, 2$, and 3, and u are respectively the components and magnitude of $\mathbf{u}$. The invariance requirements given by equation (2.2) can be satisfied by choosing

$$\hat{f}(\mathbf{y},\mathbf{u},\tau)=n(\mathbf{y},\tau)\left[\frac{m}{2\pi kT(\mathbf{y},\tau)}\right]^{3/2}\exp\left[-\frac{m|\mathbf{u}-\mathbf{q}(\mathbf{y},\tau)|^2}{2kT(\mathbf{y},\tau)}\right], \tag{2.3}$$

the local Maxwellian distribution. Here m is the particle mass and k is the Boltzmann constant. In addition

$$\begin{bmatrix} n(\mathbf{y},\tau) \\ n(\mathbf{y},\tau)\,\mathbf{q}(\mathbf{y},\tau) \\ 3n(\mathbf{y},\tau)\,kT(\mathbf{y},\tau)\end{bmatrix}=\int f(\mathbf{y},\mathbf{u},\tau)\begin{bmatrix}1\\ \mathbf{u}\\ m|\mathbf{u}-\mathbf{q}(\mathbf{y},\tau)|^2\end{bmatrix}\mathrm{d}^3u \tag{2.4}$$

defines the local number density $n(\mathbf{y},\tau)$, the fluid velocity $\mathbf{q}(\mathbf{y},\tau)$, and the absolute temperature $T(\mathbf{y},\tau)$.

It is not difficult to demonstrate that the model given by equations (2.1), (2.3), and (2.4) admits an H theorem, such that

$$\frac{\partial}{\partial\tau}\int f(\mathbf{u},\tau)\ln f(\mathbf{u},\tau)\,\mathrm{d}^3u\leq 0, \tag{2.5}$$

for spatially uniform conditions. Thus the model possesses many of the important properties of the full Boltzmann equation.

Because of equation (2.4), the model is described by a nonlinear functional equation; however, we consider circumstances for which the particle distribution function

$f(\mathbf{y}, \mathbf{u}, \tau)$ differs only slightly from an initial Maxwellian distribution $f_0(\mathbf{u})$ characterized by a set of constant initial values of the number density n_0, fluid velocity $\mathbf{q}_0$, and temperature T_0. If we now write

$$f(\mathbf{y}, \mathbf{u}, \tau) = f_0(\mathbf{u}) + f_1(\mathbf{y}, \mathbf{u}, \tau), \tag{2.6}$$

and truncate $\hat{f}(\mathbf{y}, \mathbf{u}, \tau)$ at the linear terms in a Taylor series expansion about f_0, we find that equation (2.1) can be approximated by

$$\left[\frac{\partial}{\partial t} + (\mathbf{c}+\mathbf{v})\cdot\nabla + 1\right]\hat{h}(\mathbf{x}, \mathbf{c}, t) = \int \hat{h}(\mathbf{x}, \mathbf{c}', t)K(\mathbf{c}' : \mathbf{c})e^{-c'^2}\,d^3c', \tag{2.7}$$

with

$$K(\mathbf{c}' : \mathbf{c}) = \frac{1}{\pi^{3/2}}\left[1 + 2\mathbf{c}\cdot\mathbf{c}' + \frac{2}{3}\left(c^2 - \frac{3}{2}\right)\left(c'^2 - \frac{3}{2}\right)\right], \tag{2.8}$$

and where

$$\mathbf{x} = \nu\left(\frac{m}{2kT_0}\right)^{1/2}\mathbf{y}, \qquad t = \nu\tau, \tag{2.9a,b}$$

$$\mathbf{c} = \left(\frac{m}{2kT_0}\right)^{1/2}(\mathbf{u} - \mathbf{q}_0), \qquad \mathbf{v} = \left(\frac{m}{2kT_0}\right)^{1/2}\mathbf{q}_0, \tag{2.10a,b}$$

and

$$f_0(\mathbf{c})\hat{h}(\mathbf{x}, \mathbf{c}, t) = f_1(\mathbf{x}, \mathbf{c}, t). \tag{2.11}$$

A model equation more general than equation (2.7) may be constructed, as suggested by Gross and Jackson[13] and Sirovich[27], by expanding the kernel of a linearized Boltzmann equation in an appropriately chosen complete and orthonormal set of eigenfunctions. We shall, however, restrict our attention to the linearized model described by equation (2.7).

If we now let $\phi_\alpha(\mathbf{c})$, $\alpha = 1, 2, \ldots, 5$, denote the elements of the vector

$$\boldsymbol{\phi}(c) = \frac{1}{\pi^{3/4}}\begin{bmatrix} 1 \\ \sqrt{\tfrac{2}{3}}(c^2 - \tfrac{3}{2}) \\ \sqrt{2}\,c_2 \\ \sqrt{2}\,c_3 \\ \sqrt{2}\,c_1 \end{bmatrix}, \tag{2.12}$$

where c_1, c_2, c_3, and c are respectively the components and magnitude of $\mathbf{c}$, then equation (2.8) can be written as

$$K(\mathbf{c}' : \mathbf{c}) = \tilde{\boldsymbol{\phi}}(\mathbf{c})\boldsymbol{\phi}(\mathbf{c}'). \tag{2.13}$$

Here the superscript tilde is used to denote the transpose operation. We note that the elements of $\boldsymbol{\phi}(\mathbf{c})$ obey the orthonormal conditions

$$(\phi_\alpha, \phi_\beta)_a = \delta_{\alpha,\beta}:\ \alpha,\beta = 1, 2, \ldots, 5, \tag{2.14}$$

in a Hilbert space (a) of the functions of $\mathbf{c}$ defined by the inner product

$$(A_1, A_2)_a = \int A_1(\mathbf{c})A_2(\mathbf{c})\,\mathrm{e}^{-c^2}\,\mathrm{d}^3c. \tag{2.15}$$

The elements $\phi_\alpha(\mathbf{c})$ are, of course, related to the collisional invariants which define the $\mathbf{U}$ vector in equation (2.2), and the orthogonality conditions stated in equation (2.14) are therefore direct consequences of the invariance requirements of equation (2.2).

3. THE VECTOR KINETIC EQUATION

As stated in the Introduction, we are primarily interested in steady-state gas-kinetic problems with plane symmetry. Without loss of generality, we set $\mathbf{q}_0 = \mathbf{0}$, and thus the steady-state version of equation (2.7) for

$$h(x_1, \mathbf{c}) \triangleq \hat{h}(x_1, \mathbf{c}) - 2\pi^{-3/2}c_1(c_1, \hat{h})_a \tag{3.1}$$

becomes

$$\left[c_1\frac{\partial}{\partial x_1} + 1\right]h(x_1, \mathbf{c}) = \sum_{\alpha=1}^{4} \phi_\alpha(\mathbf{c})(\phi_\alpha, h)_a. \tag{3.2}$$

We now follow Cercignani[12] and consider the functions

$$\begin{gathered} g_1(c_2, c_3) = \pi^{-1/2}, \qquad g_2(c_2, c_3) = \pi^{-1/2}(c_2^2 + c_3^2 - 1), \\ g_3(c_2, c_3) = \left(\frac{\pi}{2}\right)^{-1/2} c_2, \quad \text{and} \quad g_4(c_2, c_3) = \left(\frac{\pi}{2}\right)^{-1/2} c_3. \end{gathered} \tag{3.3}$$

It is a straightforward matter to demonstrate that the g functions given by equations (3.3) satisfy the orthonormal conditions

$$(g_\alpha, g_\beta)_b = \delta_{\alpha,\beta},\ \alpha = 1, 2, 3, \text{and } 4, \tag{3.4}$$

in a subspace (b) of the functions of c_2 and c_3 defined by the inner product

$$(B_1, B_2)_b = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_1(c_2, c_3)B_2(c_2, c_3)\mathrm{e}^{-(c_2^2+c_3^2)}\,\mathrm{d}c_2\mathrm{d}c_3. \tag{3.5}$$

We now span the Hilbert space (b) by the subspace (c) characterized by the g_α's and a subspace (d), the orthogonal complement to (c); subsequently we expand $h(x_1, \mathbf{c})$ of equation (3.2) in the manner

$$h(x_1, \mathbf{c}) = \sum_{\alpha=1}^{4} \Psi_\alpha(x_1, c_1)g_\alpha(c_2, c_3) + \Psi_5(x_1, \mathbf{c}), \tag{3.6}$$

where $\Psi_5(x_1, \mathbf{c})$ is the component of $h(x_1, \mathbf{c})$ belonging to the subspace (d). Such an expansion yields the interesting property that the inner products $(\Psi_\alpha, \phi_\alpha)_\mu$, $\alpha = 1, 2, 3$, and 4, are simply related to the perturbations of the number density, the temperature, and the transverse components of the fluid velocity, respectively.

Substituting equation (3.6) into equation (3.2) and projecting each term onto the appropriate directions of g_α in the Hilbert space (b), we obtain

$$\left[\mu\frac{\partial}{\partial x}+1\right]\hat{\boldsymbol{\Psi}}(x,\mu)=\frac{1}{\sqrt{\pi}}\mathbf{J}(\mu)\int_{-\infty}^{\infty}\tilde{\mathbf{J}}(\mu')\hat{\boldsymbol{\Psi}}(x,\mu')\,e^{-\mu'^2}\,d\mu', \tag{3.7}$$

where $\hat{\boldsymbol{\Psi}}(x, \mu)$ is a four-element vector with components $\Psi_\alpha(x, \mu)$, $\alpha = 1, 2, \ldots, 4$, and

$$\mathbf{J}(\mu)=\begin{bmatrix}\sqrt{\tfrac{2}{3}}(\mu^2-\tfrac{1}{2}) & 1 & 0 & 0\\ \sqrt{\tfrac{2}{3}} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{3.8}$$

For convenience, we have changed the variables x_1 and c_1 to x and μ. We note that the two functions $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$, characterizing the perturbations of the number density and temperature respectively, are described by a set of two coupled integrodifferential equations. These two equations are, of course, uncoupled from the functions $\Psi_3(x, \mu)$ and $\Psi_4(x, \mu)$ which describe the perturbations of the transverse momenta.

4. ELEMENTARY SOLUTIONS OF THE TWO-VECTOR TRANSPORT EQUATION RELEVANT TO TEMPERATURE-DENSITY EFFECTS

We are interested in the steady-state, gas-kinetic effects of temperature-density variations in plane-parallel media. According to equation (3.7), the relevant coupled equations are

$$\mu\frac{\partial}{\partial x}\boldsymbol{\Psi}(x,\mu)+\boldsymbol{\Psi}(x,\mu)=\frac{1}{\sqrt{\pi}}\mathbf{Q}(\mu)\int_{-\infty}^{\infty}\tilde{\mathbf{Q}}(\mu')\boldsymbol{\Psi}(x,\mu')e^{-\mu'^2}\,d\mu', \tag{4.1}$$

where $\tilde{\mathbf{Q}}(\mu)$ is the transpose of

$$\mathbf{Q}(\mu)=\begin{bmatrix}\sqrt{\tfrac{2}{3}}(\mu^2-\tfrac{1}{2}) & 1\\ \sqrt{\tfrac{2}{3}} & 0\end{bmatrix}, \tag{4.2}$$

and $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$, which are sufficient to determine the temperature–density effects, are respectively the upper and lower entries in the two-vector $\boldsymbol{\Psi}(x, \mu)$. We should like to note that equation (4.1) is quite similar to the equation of transfer used in a related study [4, 5] of the scattering of polarized light.

Following Case [7] who introduced the method of normal modes in regard to one-speed neutron-transport theory, we search for elementary solutions to equation (4.1) of the form

$$\boldsymbol{\Psi}_\xi(x,\mu)=\mathbf{F}(\xi,\mu)e^{-x/\xi}, \tag{4.3}$$

where ξ and $\mathbf{F}(\xi, \mu)$ are the eigenvalues and eigenvectors to be determined. From equation (4.1), we obtain

$$(\xi - \mu)\mathbf{F}(\xi, \mu) = \frac{1}{\sqrt{\pi}}\xi\mathbf{Q}(\mu)\mathbf{M}(\xi) \tag{4.4}$$

where the normalization vector $\mathbf{M}(\xi)$ is given by

$$\mathbf{M}(\xi) = \int_{-\infty}^{\infty} \tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi, \mu)e^{-\mu^2}\,d\mu. \tag{4.5}$$

Equation (4.4) admits both discrete eigenvalues and a continuous spectrum. We consider first the discrete spectrum: $\xi = \eta_i$, $Im\ \eta_i \neq 0$, and solve equation (4.4) to obtain

$$\mathbf{F}(\eta_i, \mu) = \frac{1}{\sqrt{\pi}}\frac{\eta_i}{\eta_i - \mu}\mathbf{Q}(\mu)\mathbf{M}(\eta_i), \tag{4.6}$$

where η_i are the zeros (in the complex plane cut along the entire real axis) of the dispersion function

$$\Lambda(z) = \det \mathbf{\Lambda}(z). \tag{4.7}$$

Here the dispersion matrix is

$$\mathbf{\Lambda}(z) = \mathbf{I} + z\int_{-\infty}^{\infty} \mathbf{\Psi}(\mu)\frac{d\mu}{\mu - z}, \tag{4.8}$$

with $\mathbf{I}$ denoting the unit matrix and the characteristic matrix given by

$$\mathbf{\Psi}(\mu) = \frac{1}{\sqrt{\pi}}\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu)e^{-\mu^2}. \tag{4.9}$$

Further, $\mathbf{M}(\eta_i)$ is a null vector of $\mathbf{\Lambda}(\eta_i)$ such that

$$\mathbf{\Lambda}(\eta_i)\mathbf{M}(\eta_i) = \mathbf{0}. \tag{4.10}$$

The argument principle [10] may be used to show that $\Lambda(z)$ has no zeros in the finite cut plane; however, since $\Lambda(z) \sim (a/z^4) + \ldots$, for $|z|$ tending to infinity, we may deduce four 'discrete' solutions to equation (4.1). In the limit $|z| \to \infty$, we obtain from equations (4.6) and (4.10)

$$\mathbf{F}_1(\mu) = \mathbf{Q}(\mu)\begin{bmatrix}1\\0\end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix}\mu^2 - \frac{1}{2}\\1\end{bmatrix}, \text{ and } \mathbf{F}_2(\mu) = \mathbf{Q}(\mu)\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}. \tag{4.11}$$

To construct the other two solutions requires a technique discussed by Case and Zweifel [8] to split the degeneracy at infinity. The resulting solutions are

$$\mathbf{\Psi}_3(x, \mu) = (\mu - x)\sqrt{\frac{2}{3}}\begin{bmatrix}\mu^2 - \frac{1}{2}\\1\end{bmatrix}, \text{ and } \mathbf{\Psi}_4(x, \mu) = (\mu - x)\begin{bmatrix}1\\0\end{bmatrix}. \tag{4.12}$$

It should be noted that equations (4.11) are solutions to equation (4.1) and to equation (4.4) in the limit $|\xi| \to \infty$; whereas, equations (4.12) are solutions only to equation (4.1).

We now consider the continuous spectrum: $\xi = \eta$, with $Im\ \eta = 0$, and the solutions to equation (4.4) are

$$\mathbf{F}(\eta, \mu) = \frac{1}{\sqrt{\pi}}\left[\eta Pv\left(\frac{1}{\eta - \mu}\right) + \lambda^*(\eta)\delta(\eta - \mu)\right]\mathbf{Q}(\mu)\mathbf{M}(\eta), \tag{4.13}$$

where $Pv(1/x)$ denotes the Cauchy principal-value distribution, and $\delta(x)$ represents the Dirac delta distribution. Pre-multiplying equation (4.13) by $\tilde{\mathbf{Q}}(\mu)\,e^{-\mu^2}$ and integrating over all μ, we find

$$[\boldsymbol{\lambda}(\eta) - \lambda^*(\eta)\boldsymbol{\Psi}(\eta)]\mathbf{M}(\eta) = \mathbf{0}, \tag{4.14}$$

where

$$\boldsymbol{\lambda}(\eta) = \mathbf{I} + \eta P\int_{-\infty}^{\infty} \boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu - \eta}; \tag{4.15}$$

and hence from

$$\det[\boldsymbol{\lambda}(\eta) - \lambda^*(\eta)\boldsymbol{\Psi}(\eta)] = 0, \tag{4.16}$$

we obtain a quadratic equation for the function $\lambda^*(\eta)$. In general there are two solutions which we label $\lambda_1^*(\eta)$ and $\lambda_2^*(\eta)$, and thus we write the two-fold degenerate continuum solutions as

$$\mathbf{F}_\alpha(\eta, \mu) = \frac{1}{\sqrt{\pi}}\left[\eta Pv\left(\frac{1}{\eta - \mu}\right) + \lambda_\alpha^*(\eta)\delta(\eta - \mu)\right]\mathbf{Q}(\mu)\mathbf{M}_\alpha(\eta),\ \alpha = 1 \text{ or } 2,\ \eta \in (-\infty, \infty), \tag{4.17}$$

where the vectors $\mathbf{M}_\alpha(\eta)$ are to be determined by the corresponding $\lambda_\alpha^*(\eta)$, through equation (4.14).

Having established the elementary solutions, we write our general solution to equation (4.1) as

$$\boldsymbol{\Psi}(x, \mu) = \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=3}^{4} A_\alpha \boldsymbol{\Psi}_\alpha(x, \mu) + \sum_{\alpha=1}^{2}\int_{-\infty}^{\infty} A_\alpha(\eta)\mathbf{F}_\alpha(\eta, \mu)e^{-x/\eta}\, d\eta, \tag{4.18}$$

where the expansion coefficients A_α and $A_\alpha(\eta)$ are to be determined once the boundary conditions of a particular problem are specified. Although in general the integral terms in equation (4.18) may diverge for $x \neq 0$, this will not be the case when the specific problems of sections 7 and 13 are considered.

5. A FULL-RANGE EXPANSION THEOREM

To ensure that the normal modes developed in the previous sections are sufficiently general for full-range, $\mu \in (-\infty, \infty)$, boundary-value problems, we should now like to prove a basic result.

Theorem 1. *The functions* $\mathbf{F}_1(\mu)$, $\mathbf{F}_2(\mu)$, $\mathbf{F}_3(\mu) = \boldsymbol{\Psi}_3(0, \mu)$, $\mathbf{F}_4(\mu) = \boldsymbol{\Psi}_4(0, \mu)$, *and* $\mathbf{F}_\alpha(\eta, \mu)$, $\alpha = 1$ and 2, $\eta \in (-\infty, \infty)$, *form a complete basis set for the expansion of an* *arbitrary two-vector* $\mathbf{I}(\mu)$, *which is Hölder continuous on any open interval of the real axis and, for sufficiently large* $|\mu|$, satisfies

$$|I_\alpha(\mu)| \exp(-|\mu|) < \infty,\ \alpha = 1 \text{ or } 2,$$

in the sense that

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{4} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_{-\infty}^{\infty} A_\alpha(\eta)\mathbf{F}_\alpha(\eta, \mu)\,\mathrm{d}\eta,\ \mu \in (-\infty, \infty). \tag{5.1}$$

To prove the theorem, we shall construct an analytical solution to the above coupled singular integral equations. For the sake of brevity, we write

$$\hat{\mathbf{I}}(\mu) = \mathbf{I}(\mu) - \sum_{\alpha=1}^{4} A_\alpha \mathbf{F}_\alpha(\mu), \tag{5.2}$$

introduce the (2×2) matrix

$$\mathbf{G}(\eta, \mu) = [\mathbf{F}_1(\eta, \mu) \quad \mathbf{F}_2(\eta, \mu)], \tag{5.3}$$

let $\mathbf{A}(\eta)$ denote a vector with elements $A_\alpha(\eta)$, $\alpha = 1$ and 2, and thus write equation (5.1) as

$$\hat{\mathbf{I}}(\mu) = \int_{-\infty}^{\infty} \mathbf{G}(\eta, \mu)\mathbf{A}(\eta)\,\mathrm{d}\eta,\ \mu \in (-\infty, \infty). \tag{5.4}$$

Pre-multiplying equation (4.13) by $\tilde{\mathbf{Q}}(\mu)\,\mathrm{e}^{-\mu^2}$ and invoking equation (4.14), we obtain

$$\tilde{\mathbf{Q}}(\mu)\mathbf{G}(\eta, \mu)\,\mathrm{e}^{-\mu^2} = \left[\eta Pv\left(\frac{1}{\eta - \mu}\right)\mathbf{\Psi}(\mu) + \delta(\eta - \mu)\boldsymbol{\lambda}(\eta)\right]\mathbf{V}(\eta), \tag{5.5}$$

where

$$\mathbf{V}(\eta) = [\mathbf{M}_1(\eta) \quad \mathbf{M}_2(\eta)] \tag{5.6}$$

is the (2×2) normalization matrix. We now pre-multiply equation (5.4) by $\tilde{\mathbf{Q}}(\mu)\,\mathrm{e}^{-\mu^2}$, make use of equation (5.5), and integrate the δ term to obtain

$$\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\mathrm{e}^{-\mu^2} = \boldsymbol{\lambda}(\mu)\mathbf{B}(\mu) + \mathbf{\Psi}(\mu)P\int_{-\infty}^{\infty} \eta\mathbf{B}(\eta)\frac{\mathrm{d}\eta}{\eta - \mu}, \tag{5.7}$$

where $\mathbf{B}(\eta) = \mathbf{V}(\eta)\mathbf{A}(\eta)$. Equation (5.7) may now be solved explicitly by using the theory of Muskhelishvili[21]. To convert equation (5.7) to a special form of a matrix Riemann problem, we introduce the sectionally holomorphic matrix

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \eta\mathbf{B}(\eta)\frac{\mathrm{d}\eta}{\eta - z}. \tag{5.8}$$

The boundary values of $\mathbf{N}(z)$ as z approaches the real line from above (+) and below

$(-)$ follow from the Plemelj formulae[21]:

$$\pi i[\mathbf{N}^{+}(\mu)+\mathbf{N}^{-}(\mu)] = P\int_{-\infty}^{\infty} \eta\mathbf{B}(\eta)\frac{\mathrm{d}\eta}{\eta-\mu} \tag{5.9a}$$

and

$$\mathbf{N}^{+}(\mu)-\mathbf{N}^{-}(\mu) = \mu\mathbf{B}(\mu). \tag{5.9b}$$

In a similar manner, the boundary values of the dispersion matrix follow from equation (4.8):

$$\mathbf{\Lambda}^{+}(\mu)+\mathbf{\Lambda}^{-}(\mu) = 2\boldsymbol{\lambda}(\mu), \tag{5.10a}$$

and

$$\mathbf{\Lambda}^{+}(\mu)-\mathbf{\Lambda}^{-}(\mu) = 2\pi i\mu\mathbf{\Psi}(\mu). \tag{5.10b}$$

Equations (5.9) and (5.10) may now be used in equation (5.7) to yield

$$\mu\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\mathrm{e}^{-\mu^2} = \mathbf{\Lambda}^{+}(\mu)\mathbf{N}^{+}(\mu)-\mathbf{\Lambda}^{-}(\mu)\mathbf{N}^{-}(\mu), \tag{5.11}$$

which can be solved to give

$$\mathbf{N}(z) = \mathbf{\Lambda}^{-1}(z)\frac{1}{2\pi i}\int_{-\infty}^{\infty} \mu\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\mathrm{e}^{-\mu^2}\frac{\mathrm{d}\mu}{\mu-z}. \tag{5.12}$$

We note that for large $|z|$,

$$\mathbf{\Lambda}(z) \sim -\frac{1}{z^2}\begin{bmatrix} \frac{7}{6} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}, \text{ as } |z|\to\infty, \tag{5.13}$$

and

$$\mathbf{\Lambda}^{-1}(z) \sim -\frac{12}{5}z^2\begin{bmatrix} \frac{1}{2} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & \frac{7}{6} \end{bmatrix}, \text{ as } |z|\to\infty, \tag{5.14}$$

and therefore if the $\mathbf{N}(z)$ as given by equation (5.12) is to vanish when $|z|$ tends to infinity, as equation (5.8) prescribes, we must impose on the vector $\hat{\mathbf{I}}(\mu)$ the four constraints

$$\int_{-\infty}^{\infty} \mu^{\alpha}\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu = \mathbf{0}, \quad \alpha = 1 \text{ and } 2. \tag{5.15}$$

Recalling equation (5.2) for $\hat{\mathbf{I}}(\mu)$, we observe that equation (5.15) will be inherently satisfied if we specify the expansion coefficients A_α, $\alpha = 1, 2, 3$, and 4, to be

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \frac{12}{5\sqrt{6\pi}} \begin{bmatrix} -\frac{3}{2}I_{12} + I_{14} + I_{22} \\ \frac{8}{\sqrt{6}}I_{12} - \sqrt{\frac{2}{3}}I_{14} - \sqrt{\frac{2}{3}}I_{22} \\ -\frac{3}{2}I_{11} + I_{13} + I_{21} \\ \frac{8}{\sqrt{6}}I_{11} - \sqrt{\frac{2}{3}}I_{13} - \sqrt{\frac{2}{3}}I_{21} \end{bmatrix}, \tag{5.16}$$

where

$$I_{\alpha\beta} \triangleq \int_{-\infty}^{\infty} \mu^\beta I_\alpha(\mu) e^{-\mu^2}\, d\mu, \quad \alpha = 1 \text{ or } 2;\ \beta = 1, 2, 3 \text{ or } 4, \tag{5.17}$$

and $I_1(\mu)$ and $I_2(\mu)$ are respectively the upper and lower entries of $\mathbf{I}(\mu)$. Theorem 1 is therefore established.

Although we could pursue equations (5.9b) and (5.12) to obtain explicit results for the continuum coefficients $A_\alpha(\eta)$, $\alpha = 1$ or 2, we prefer to summarize the final expressions in terms of the formalism of the full-range orthogonality relations given in the next section.

6. ORTHOGONALITY RELATIONS AND EXPLICIT SOLUTIONS

We should first like to state the general orthogonality relation relevant to all solutions including the special distributions, $\mathbf{F}(\xi, \mu)$, of the separated equation (4.4).

Theorem 2. *All eigenvectors* $\mathbf{F}(\xi, \mu)$ *which are solutions of equation* (4.4) *are orthogonal on the full range,* $\mu \in (-\infty, \infty)$, *in the sense that*

$$\int_{-\infty}^{\infty} \tilde{\mathbf{F}}(\xi', \mu)\mathbf{F}(\xi, \mu) e^{-\mu^2} \mu\, d\mu = 0, \quad \xi' \neq \xi. \tag{6.1}$$

To prove the theorem, equation (4.4) is first pre-multiplied by $\tilde{\mathbf{F}}(\xi', \mu)e^{-\mu^2}/\xi$, the transpose of equation (4.4) with ξ changed to ξ' is post-multiplied by $\mathbf{F}(\xi, \mu)e^{-\mu^2}/\xi'$, and the two resulting equations are then integrated over all μ and subtracted one from the other to yield

$$\left(\frac{1}{\xi} - \frac{1}{\xi'}\right) \int_{-\infty}^{\infty} \tilde{\mathbf{F}}(\xi', \mu)\mathbf{F}(\xi, \mu) e^{-\mu^2} \mu\, d\mu = 0, \tag{6.2}$$

which establishes equation (6.1). Though equation (6.2) is a general statement of full-range orthogonality, it is clear that several additional relations are required here. First of all, since $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ are both associated with $\xi \to \infty$, equation (6.2) does not ensure that they will be mutually orthogonal in the sense of equation (6.1). In addition, the vectors $\mathbf{F}_3(\mu)$ and $\mathbf{F}_4(\mu)$, being derived from the solutions of equation (4.1), rather than equation (4.4), are not included in Theorem 2. However, it can be easily shown that $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ are mutually orthogonal, and, in fact, self-orthogonal; the same is true for $\mathbf{F}_3(\mu)$ and $\mathbf{F}_4(\mu)$. In addition, $\mathbf{F}_3(\mu)$ and $\mathbf{F}_4(\mu)$ are orthogonal to the continuum solutions $\mathbf{F}_1(\eta, \mu)$ and $\mathbf{F}_2(\eta, \mu)$. We note that $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ are not orthogonal to $\mathbf{F}_3(\mu)$ and $\mathbf{F}_4(\mu)$; however, suitably defined adjoint vectors for these special cases can be developed by employing a Schmidt-type procedure.

Considering first the normalization integrals related to the solutions given by equation (4.13), we find

$$\int_{-\infty}^{\infty} \tilde{\mathbf{F}}_\alpha(\eta', \mu)\mathbf{F}_\beta(\eta, \mu)e^{-\mu^2}\mu\, d\mu = S(\eta)\delta(\eta - \eta')\delta_{\alpha,\beta};\ \alpha,\beta = 1, 2, \tag{6.3}$$

where

$$S(\eta) = \frac{\eta}{\sqrt{\pi}}[\pi^2\eta^2 + \lambda_\alpha^*(\eta)\lambda_\beta^*(\eta)]\tilde{\mathbf{M}}_\alpha(\eta)\boldsymbol{\Psi}(\eta)\mathbf{M}_\beta(\eta). \tag{6.4}$$

The Kronecker $\delta_{\alpha,\beta}$ appearing in equation (6.3) should be noted since it ensures that the degenerate continuum solutions given by equation (4.13) are orthogonal even for $\eta' = \eta$. To establish equation (6.3) requires the use of the Poincaré-Bertrand formula [21] and

$$[\lambda_\alpha^*(\eta) - \lambda_\beta^*(\eta)]\tilde{\mathbf{M}}_\alpha(\eta)\boldsymbol{\Psi}(\eta)\mathbf{M}_\beta(\eta) = 0, \tag{6.5}$$

a relation which can be deduced from equation (4.14).

Though the representations of the two continuum solutions given by equation (4.13) were convenient for proving the full-range expansion theorem, we choose to make use of more explicit forms for actual applications. We note that equation (4.16) is quadratic in $\lambda^*(\eta)$, and thus the two solutions will in general involve radicals. To avoid the cumbersome nature of the ensuing solutions, we prefer the linear combinations

$$\boldsymbol{\Phi}_\alpha(\eta, \mu) = T_{\alpha 1}(\eta)\mathbf{F}_1(\eta, \mu) + T_{\alpha 2}(\eta)\mathbf{F}_2(\eta, \mu), \quad \alpha = 1 \text{ and } 2, \tag{6.6}$$

which, for judicious choices of $T_{\alpha\beta}(\eta)$, enable us to deduce the more tractable solutions

$$\boldsymbol{\Phi}_1(\eta, \mu) = \begin{bmatrix} \frac{1}{\sqrt{\pi}}\eta\left(\mu^2 - \frac{1}{2}\right)Pv\left(\frac{1}{\eta - \mu}\right)e^{-\eta^2} + \lambda_1(\eta)\delta(\eta - \mu) \\ \frac{1}{\sqrt{\pi}}\eta Pv\left(\frac{1}{\eta - \mu}\right)e^{-\eta^2} + \left[\frac{1}{2} + \lambda_0(\eta)\right]\delta(\eta - \mu) \end{bmatrix} \tag{6.7a}$$

and

$$\boldsymbol{\Phi}_2(\eta, \mu) = \begin{bmatrix} \frac{1}{\sqrt{\pi}}\eta Pv\left(\frac{1}{\eta - \mu}\right)e^{-\eta^2} + \lambda_0(\eta)\delta(\eta - \mu) \\ \frac{1}{2}\delta(\eta - \mu) \end{bmatrix}, \tag{6.7b}$$

where

$$\lambda_0(\eta) = 1 + \frac{1}{\sqrt{\pi}}\eta P\int_{-\infty}^{\infty} e^{-\mu^2}\frac{d\mu}{\mu - \eta}, \tag{6.8a}$$

or

$$\lambda_0(\eta) = 1 - 2\eta\, e^{-\eta^2} \int_0^\eta e^{\mu^2}\, d\mu, \tag{6.8b}$$

and

$$\lambda_1(\eta) = \tfrac{1}{2} + (\eta^2 - \tfrac{1}{2})\lambda_0(\eta). \tag{6.9}$$

We note that equations (6.7) are not mutually orthogonal for $\eta' = \eta$; however, a Schmidt-type procedure may be used here as well. Since our final adjoint vectors follow in a manner analogous to that reported by Siewert and Zweifel [26], we shall simply summarize our conclusions below. For the case of the degenerate continuum modes, we find that the procedure discussed in reference [26] can be used to establish the required adjoint vectors. To unify our notation, we also define

$$\boldsymbol{\Phi}_\alpha(\mu) \triangleq \mathbf{F}_\alpha(\mu), \quad \alpha = 1, 2, 3 \text{ and } 4. \tag{6.10}$$

The *orthonormal* full-range adjoint set is given by:

$$\mathbf{X}_1(\mu) = \frac{1}{5\sqrt{\pi}}[6\boldsymbol{\Phi}_3(\mu) - 2\sqrt{6}\boldsymbol{\Phi}_4(\mu)], \tag{6.11a}$$

$$\mathbf{X}_2(\mu) = \frac{1}{5\sqrt{\pi}}[-2\sqrt{6}\boldsymbol{\Phi}_3(\mu) + 14\boldsymbol{\Phi}_4(\mu)], \tag{6.11b}$$

$$\mathbf{X}_3(\mu) = \frac{1}{5\sqrt{\pi}}[6\boldsymbol{\Phi}_1(\mu) - 2\sqrt{6}\boldsymbol{\Phi}_2(\mu)], \tag{6.11c}$$

$$\mathbf{X}_4(\mu) = \frac{1}{5\sqrt{\pi}}[-2\sqrt{6}\boldsymbol{\Phi}_1(\mu) + 14\boldsymbol{\Phi}_2(\mu)], \tag{6.11d}$$

$$\mathbf{X}_1(\eta, \mu) = \frac{1}{N(\eta)}[N_{22}(\eta)\boldsymbol{\Phi}_1(\eta, \mu) - N_{12}(\eta)\boldsymbol{\Phi}_2(\eta, \mu)], \tag{6.11e}$$

and

$$\mathbf{X}_2(\eta, \mu) = \frac{1}{N(\eta)}[N_{11}(\eta)\boldsymbol{\Phi}_2(\eta, \mu) - N_{12}(\eta)\boldsymbol{\Phi}_1(\eta, \mu)], \tag{6.11f}$$

where

$$N_{11}(\eta) = [\lambda_0(\eta) + \tfrac{1}{2}]^2 + \lambda_1^2(\eta) + \pi\eta^2[(\eta^2 - \tfrac{1}{2})^2 + 1]\, e^{-2\eta^2},$$

$$N_{12}(\eta) = \lambda_0(\eta)\lambda_1(\eta) + \tfrac{1}{2}\lambda_0(\eta) + \tfrac{1}{4} + \pi\eta^2(\eta^2 - \tfrac{1}{2})\, e^{-2\eta^2},$$

$$N_{22}(\eta) = \lambda_0^2(\eta) + \tfrac{1}{4} + \pi\eta^2\, e^{-2\eta^2},$$

and

$$N(\eta) = \tfrac{9}{4}\eta\, e^{-\eta^2}\Lambda^+(\eta)\Lambda^-(\eta). \tag{6.12}$$

The required orthogonality relations among the full-range basis and adjoint sets are:

$$\int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\mu)\boldsymbol{\Phi}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\mu\,\mathrm{d}\mu = \delta_{\alpha,\beta};\ \alpha,\beta = 1, 2, 3, \text{or } 4, \tag{6.13a}$$

$$\int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\mu)\boldsymbol{\Phi}_\beta(\eta,\mu)\,\mathrm{e}^{-\mu^2}\mu\,\mathrm{d}\mu = 0;\ \alpha = 1, 2, 3, \text{or } 4,\ \beta = 1 \text{ or } 2, \tag{6.13b}$$

$$\int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\eta',\mu)\boldsymbol{\Phi}_\beta(\eta,\mu)\,\mathrm{e}^{-\mu^2}\mu\,\mathrm{d}\mu = \delta(\eta-\eta')\delta_{\alpha,\beta};\ \alpha,\beta = 1 \text{ or } 2, \tag{6.13c}$$

$$\int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\eta',\mu)\boldsymbol{\Phi}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\mu\,\mathrm{d}\mu = 0;\ \alpha = 1 \text{ or } 2,\ \beta = 1, 2, 3, \text{or } 4. \tag{6.13d}$$

With the formalism thus established, we note that all expansion coefficients in equations of the form

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{4} A_\alpha\boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=1}^{2}\int_{-\infty}^{\infty} A_\alpha(\eta)\boldsymbol{\Phi}_\alpha(\eta,\mu)\,\mathrm{d}\eta,\ \mu\in(-\infty,\infty), \tag{6.14}$$

may be expressed immediately in terms of inner products:

$$A_\alpha = \int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\mu\,\mathrm{d}\mu,\ \alpha = 1, 2, 3 \text{ and } 4, \tag{6.15}$$

and

$$A_\alpha(\eta) = \int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\eta,\mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\mu\,\mathrm{d}\mu,\ \alpha = 1 \text{ and } 2. \tag{6.16}$$

7. THE INFINITE-MEDIUM GREEN'S FUNCTION

In order to illustrate the use of the elementary solutions of equation (4.1) and the relevant orthogonality relations, we should now like to develop the infinite-medium Green's function. Here we seek a solution to

$$\mu\frac{\partial}{\partial x}\boldsymbol{\Psi}(x,\mu) + \boldsymbol{\Psi}(x,\mu) = \frac{1}{\sqrt{\pi}}\mathbf{Q}(\mu)\int_{-\infty}^{\infty}\tilde{\mathbf{Q}}(\mu')\boldsymbol{\Psi}(x,\mu')\,\mathrm{e}^{-\mu'^2}\mathrm{d}\mu' + \mathbf{S}(x_0,\mu_1,\mu_2;x,\mu), \tag{7.1}$$

where

$$\mathbf{S}(x_0,\mu_1,\mu_2;x,\mu) = \delta(x-x_0)\begin{bmatrix}\rho_1\,\delta(\mu-\mu_1)\\ \rho_2\,\delta(\mu-\mu_2)\end{bmatrix}. \tag{7.2}$$

Clearly, since the kinetic equation conserves particles, kinetic energy, and momentum, there will exist no bounded (at infinity) solution to equation (7.1); however, the Green's function we develop may be used in the classical manner to construct particular solutions to equation (7.1) for arbitrary inhomogeneous source terms for semi-infinite or finite media. As discussed by Case and Zweifel [8], we neglect the inhomogeneous term in equation (7.1) and require the solution to the resulting homogeneous equation to satisfy the 'jump' boundary condition

$$\mu[\boldsymbol{\Psi}(x_0, \mu_1, \mu_2; x_0^+, \mu) - \boldsymbol{\Psi}(x_0, \mu_1, \mu_2; x_0^-, \mu)] = \begin{bmatrix} \rho_1\ \delta(\mu - \mu_1) \\ \rho_2\ \delta(\mu - \mu_2) \end{bmatrix}, \quad \mu \in (-\infty, \infty), \tag{7.3}$$

where the argument list has been extended to include the parameters x_0, μ_1, and μ_2. We therefore write the desired solution as

$$\boldsymbol{\Psi}(x_0, \mu_1, \mu_2; x, \mu) = \sum_{\alpha=1}^{2} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_0^\infty A_\alpha(\eta) \boldsymbol{\Phi}_\alpha(\eta, \mu)\, e^{-(x-x_0)/\eta}\, d\eta, \tag{7.4a}$$

for $x > x_0$,

and

$$\boldsymbol{\Psi}(x_0, \mu_1, \mu_2; x, \mu) = -\sum_{\alpha=3}^{4} A_\alpha \boldsymbol{\Psi}_\alpha(x - x_0, \mu) - \sum_{\alpha=1}^{2} \int_{-\infty}^{0} A_\alpha(\eta) \boldsymbol{\Phi}_\alpha(\eta, \mu)\, e^{-(x-x_0)/\eta}\, d\eta$$

for $x < x_0$. (7.4b)

Substitution of equations (7.4) into equation (7.3) yields the full-range expansion

$$\begin{bmatrix} \rho_1\ \delta(\mu - \mu_1) \\ \rho_2\ \delta(\mu - \mu_2) \end{bmatrix} = \mu \left\{ \sum_{\alpha=1}^{4} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_{-\infty}^{\infty} A_\alpha(\eta) \boldsymbol{\Phi}_\alpha(\eta, \mu)\, d\eta \right\}, \mu \in (-\infty, \infty). \tag{7.5}$$

Though the left-hand side of equation (7.5) certainly is not a Hölder function, Case and Zweifel[8] have concluded that expansion theorems similar to our Theorem 1 remain valid formally even for this type of delta distribution. We therefore pre-multiply equation (7.5) by $\tilde{\mathbf{X}}_\alpha(\mu)\, e^{-\mu^2}$, $\alpha = 1, 2, 3$, or 4 and $\tilde{\mathbf{X}}_\alpha(\eta', \mu)\, e^{-\mu^2}$, $\alpha = 1$ or 2, and integrate over all μ to find, after invoking equation (6.13),

$$A_\alpha = \rho_1 X_{\alpha 1}(\mu_1)\, e^{-\mu_1^2} + \rho_2 X_{\alpha 2}(\mu_2)\, e^{-\mu_2^2}, \alpha = 1, 2, 3, \text{and } 4, \tag{7.6a}$$

and

$$A_\alpha(\eta) = \rho_1 X_{\alpha 1}(\eta, \mu_1)\, e^{-\mu_1^2} + \rho_2 X_{\alpha 2}(\eta, \mu_2)\, e^{-\mu_2^2}, \alpha = 1 \text{ and } 2, \tag{7.6b}$$

where the subscripts 1 and 2 are used to denote the upper and lower elements of the $\mathbf{X}$ vectors. Since all expansion coefficients required in equations (7.4) are given by equations (7.6), the infinite-medium Green's function is established.

8. A HALF-RANGE EXPANSION THEOREM

Having developed in sections 5 and 6 the necessary completeness and orthogonality properties of our normal modes, we should now like to discuss the analysis required for the considerably more interesting problems defined by half-range, $\mu \in (0, \infty)$, boundary conditions. The following theorem states the very important half-range expansion properties basic to a certain subset of our derived elementary solutions.

Theorem 3. *The functions* $\mathbf{F}_1(\mu)$, $\mathbf{F}_2(\mu)$ *and* $\mathbf{F}_\alpha(\eta, \mu)$, $\alpha = 1$ *and* 2, $\eta \in (0, \infty)$, *form a complete basis set for the expansion of an arbitrary two-vector* $\mathbf{I}(\mu)$ *which is Hölder* *continuous on any open interval of the positive real axis and, for sufficiently large* $|\mu|$, satisfies

$$|\mathbf{I}_\alpha(\mu)| \exp(-|\mu|) < \infty, \ \alpha = 1 \text{ or } 2,$$

in the sense that

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_0^\infty A_\alpha(\eta)\mathbf{F}_\alpha(\eta, \mu)\, \mathrm{d}\eta, \ \mu \in (0, \infty). \tag{8.1}$$

To prove this theorem, we premultiply equation (8.1) by $e^{-\mu^2} \tilde{\mathbf{Q}}(\mu)$, integrate the δ term and use equations (4.13) and (4.14) to obtain

$$\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\, e^{-\mu^2} = \boldsymbol{\lambda}(\mu)\mathbf{B}(\mu) + \boldsymbol{\Psi}(\mu) P \int_0^\infty \eta \mathbf{B}(\eta) \frac{\mathrm{d}\eta}{\eta - \mu}, \ \mu \in (0, \infty), \tag{8.2}$$

where

$$\hat{\mathbf{I}}(\mu) = \mathbf{I}(\mu) - \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) \tag{8.3}$$

and

$$\mathbf{B}(\eta) = \mathbf{V}(\eta)\mathbf{A}(\eta). \tag{8.4}$$

In addition, $\mathbf{V}(\eta)$ is given by equation (5.6) and the unknown $\mathbf{A}(\eta)$ has elements $A_1(\eta)$ and $A_2(\eta)$. In a manner similar to that used to prove Theorem 1, we now introduce

$$\mathbf{N}(z) = \frac{1}{2\pi i} \int_0^\infty \eta \mathbf{B}(\eta) \frac{\mathrm{d}\eta}{\eta - z}. \tag{8.5}$$

The $\mathbf{N}$ matrix is clearly analytic in the complex plane cut along the positive real axis. Further, the Plemelj formulae[21] can be used, with equation (8.5), to show that the boundary values of $\mathbf{N}(z)$ satisfy

$$\pi i [\mathbf{N}^+(\mu) + \mathbf{N}^-(\mu)] = P \int_0^\infty \eta \mathbf{B}(\eta) \frac{\mathrm{d}\eta}{\eta - \mu} \tag{8.6a}$$

and

$$\mathbf{N}^+(\mu) - \mathbf{N}^-(\mu) = \mu \mathbf{B}(\mu). \tag{8.6b}$$

Equations (8.6) can now be used, along with equations (5.10), to express equation (8.2) in the form

$$\mu \tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\, e^{-\mu^2} = \boldsymbol{\Lambda}^+(\mu)\mathbf{N}^+(\mu) - \boldsymbol{\Lambda}^-(\mu)\mathbf{N}^-(\mu), \ \mu \in (0, \infty). \tag{8.7}$$

If we now let $\tilde{\mathbf{X}}(z)$ denote a canonical (non-singular in the finite plane) solution to the homogeneous Riemann problem defined by

$$\tilde{\mathbf{X}}^+(\mu) = \mathbf{G}(\mu)\tilde{\mathbf{X}}^-(\mu), \ \mu \in (0, \infty), \tag{8.8a}$$

where

$$\mathbf{G}(\mu) = \mathbf{\Lambda}^{+}(\mu)[\mathbf{\Lambda}^{-}(\mu)]^{-1}, \tag{8.8b}$$

then equation (8.7) can be solved immediately to yield

$$\mathbf{N}(z) = \mathbf{X}^{-1}(z)\left[\frac{1}{2\pi i}\int_0^\infty \mathbf{\Gamma}(\mu)\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\,\mathrm{e}^{-\mu^2}\frac{\mathrm{d}\mu}{\mu - z} + \mathbf{P}(z)\right]. \tag{8.9}$$

Here, $\mathbf{P}(z)$ is a matrix of polynomials, and

$$\mathbf{\Gamma}(\mu) = \mu\mathbf{X}^{+}(\mu)[\mathbf{\Lambda}^{+}(\mu)]^{-1}. \tag{8.10}$$

Since the $\mathbf{G}$ matrix given by equation (8.8b) is continuous for $\mu \in [0, \infty)$, $\mathbf{G}(0) = \mathbf{I}$ and $\mathbf{G}(\mu) \to \mathbf{I}$ as $\mu \to \infty$, the analysis of Mandžavidze and Hvedelidze[20] can be used, after an elementary transformation of variables, to ensure the existence of a canonical solution to the Riemann problem defined by equation (8.8a). In section 9 we argue that the partial indices κ_1 and κ_2 associated with our canonical solution $\tilde{\mathbf{X}}(z)$ are

$$\kappa_1 = \kappa_2 = 1, \tag{8.11}$$

and thus if we allow our canonical matrix $\tilde{\mathbf{X}}(z)$ to be of normal form at infinity[21], we may write

$$\lim_{z\to\infty} z\,\mathbf{X}(z) = \mathbf{\Delta}, \tag{8.12}$$

where $\mathbf{\Delta}$ is nonsingular and bounded.

From the defining equation (8.5), we observe that $z\,\mathbf{N}(z)$ must be bounded as $|z| \to \infty$, and thus from equation (8.9) we conclude that

$$\lim_{|z|\to\infty} z\mathbf{\Delta}^{-1}\left[\frac{-1}{2\pi i}\int_0^\infty \mathbf{\Gamma}(\mu)\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu + z\mathbf{P}(z)\right] < \infty; \tag{8.13}$$

we must therefore set $\mathbf{P}(z) = \mathbf{0}$ and, in addition, require that

$$\int_0^\infty \mathbf{\Gamma}(\mu)\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu = \mathbf{0}. \tag{8.14}$$

Equation (8.14) is, of course, not satisfied by all $\hat{\mathbf{I}}(\mu)$, but recalling equation (8.3), we conclude that choosing the discrete expansion coefficients to be solutions of

$$\int_0^\infty \mathbf{\Gamma}(\mu)\mathbf{\Psi}(\mu)\,\mathrm{d}\mu\begin{bmatrix}A_1\\A_2\end{bmatrix} = \frac{1}{\sqrt{\pi}}\int_0^\infty \mathbf{\Gamma}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu \tag{8.15}$$

renders equation (8.1) valid for all appropriate $\mathbf{I}(\mu)$. The matrix in equation (8.15) whose inverse is required to obtain A_1 and A_2 can be shown to be non-singular by making use of a Cauchy integral representation of $\mathbf{X}(z)$. The theorem is therefore proved.

Though, as for the full-range case, we could pursue this completeness proof to construct the continuum expansion coefficients $\mathbf{A}(\eta)$, $\eta \in (0, \infty)$, we find it more convenient to express the final results in terms of half-range orthogonality relations.

9. A PROOF REGARDING THE PARTIAL INDICES OF THE RIEMANN PROBLEM

The proof of the half-range expansion theorem given in section 8 is based on the proposition that the partial indices κ_1 and κ_2 are both non-negative. In fact, equation (8.13) is valid only if the partial indices are given by $\kappa_1 = \kappa_2 = 1$. In this section we develop the required proof that $\kappa_1 = \kappa_2 = 1$.

We consider then the homogeneous matrix Riemann problem defined by

$$\boldsymbol{\Phi}^{+}(\mu) = \mathbf{G}(\mu)\boldsymbol{\Phi}^{-}(\mu),\ \mu \in [0, \infty), \tag{9.1}$$

where

$$\mathbf{G}(\mu) = \boldsymbol{\Lambda}^{+}(\mu)[\boldsymbol{\Lambda}^{-}(\mu)]^{-1},\ \mu \in [0, \infty). \tag{9.2}$$

Here we seek a matrix $\boldsymbol{\Phi}(z)$ analytic in the plane cut along the positive real axis, non-singular in the finite plane, and with boundary values $\boldsymbol{\Phi}^{\pm}(\mu)$ which satisfy equation (9.1).

Since $\mathbf{G}(0) = \mathbf{I}$ and $\mathbf{G}(\mu) \to \mathbf{I}$ in the limit as $\mu \to \infty$, we can define $\mathbf{G}(\mu) = \mathbf{I}$ on the entire negative real axis and thus consider equation (9.1) for $\mu \in (-\infty, \infty)$. To make use of the results developed by Mandžavidze and Hvedelidze[20], valid for closed contours, we make the change of variables

$$\zeta = \frac{i - z}{i + z} \tag{9.3}$$

which maps the upper half of the z plane into the interior of the unit circle in the ζ-plane. We note that the positive (negative) real axis maps into $|\zeta| = 1$, $Im\ \zeta > (<)\ 0$. The existence of a solution to the Riemann problem in the ζ plane follows from the theory of Mandžavidze and Hvedelidze[20], since the resulting $\mathbf{G}$ matrix is continuous on the unit circle, and $\boldsymbol{\Phi}(z)$, the canonical solution in the z-plane, is the image of the solution in the ζ plane postmultiplied by an appropriate matrix of rational functions.

It can be demonstrated[5] that the $\boldsymbol{\Lambda}$ matrix can be factored as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Phi}(z)\mathscr{P}(z)\tilde{\boldsymbol{\Phi}}(-z) \tag{9.4}$$

where $\boldsymbol{\Phi}(z)$ is any canonical solution (of ordered normal form at infinity) to equation (9.1) and $\mathscr{P}(z)$ is a matrix of polynomials, which depends on the particular choice for $\boldsymbol{\Phi}(z)$.

The fact that $\overline{\mathbf{G}(\mu)} = [\mathbf{G}(\mu)]^{-1}$, where the bar indicates the complex conjugate, enables us to extend the results of Siewert and Burniston's[25] Theorem II to the Riemann problem defined by equation (9.1):

Theorem 4. *There exists at least one canonical matrix $\boldsymbol{\Phi}_1(z)$ of ordered normal form at infinity for the Riemann problem defined by equation* (9.1) *such that $\overline{\boldsymbol{\Phi}_1(\bar{z})} = \boldsymbol{\Phi}_1(z)$.*

Since the proof of Theorem 4 follows very closely one previously reported[25], it will not be given here.

If we use $\boldsymbol{\Phi}_1(z)$ in the factorization of $\boldsymbol{\Lambda}(z)$, the resulting polynomial matrix $\mathscr{P}(z)$ is such that $\mathscr{P}(z) = \tilde{\mathscr{P}}(-z)$ and $\mathscr{P}(z) = \overline{\mathscr{P}(\bar{z})}$, since $\boldsymbol{\Lambda}(z) = \tilde{\boldsymbol{\Lambda}}(z) = \boldsymbol{\Lambda}(-z)$ and $\boldsymbol{\Lambda}(z) = \overline{\boldsymbol{\Lambda}(\bar{z})}$.

By definition[21], a canonical solution of ordered normal form at infinity is such that

$$\lim_{|z|\to\infty} \boldsymbol{\Phi}_1(z)\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} = \mathbf{K},\ \det \mathbf{K} \neq 0, \tag{9.5}$$

where κ_1 and $\kappa_2 \geq \kappa_1$ are the partial indices. Furthermore the sum of κ_1 and κ_2 must yield the total index κ, which in the manner of Muskelishvili[21] can be computed directly once the $\mathbf{G}$ matrix and the appropriate contour are specified. For this problem, we find

$$\kappa_1 + \kappa_2 = \kappa = 2. \tag{9.6}$$

If we now evaluate equation (9.4), for $\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}_1(z)$ at $z = 0$, we obtain

$$\mathscr{P}(0) = \boldsymbol{\Phi}_1^{-1}(0)\tilde{\boldsymbol{\Phi}}_1^{-1}(0), \tag{9.7}$$

and since $\boldsymbol{\Phi}_1(0)$ is real (recall that $\boldsymbol{\Phi}_1(z) = \overline{\boldsymbol{\Phi}_1(\bar{z})}$), we conclude from equation (9.7) that $\mathscr{P}_{11}(0) \neq 0$ and $\mathscr{P}_{22}(0) \neq 0$. Again from equation (9.4), for $\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}_1(z)$, we can write, after using equation (5.13),

$$\mathscr{P}(z) \to -\frac{1}{z^2}\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} \mathbf{K}^{-1}\begin{bmatrix} \frac{7}{6} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} & \frac{1}{2} \end{bmatrix}\tilde{\mathbf{K}}^{-1}\begin{bmatrix} (-z)^{\kappa_1} & 0 \\ 0 & (-z)^{\kappa_2} \end{bmatrix}, \tag{9.8}$$

$$\text{as } |z| \to \infty,$$

from which it follows, since $\mathbf{K}$ is real, that

$$\kappa_1 = \kappa_2 = 1. \tag{9.9}$$

It is clear, since $\boldsymbol{\Phi}_1(z)$ is a canonical solution of ordered normal form at infinity, and since $\kappa_1 = \kappa_2 = 1$, that

$$\boldsymbol{\Phi}_0(z) \stackrel{\Delta}{=} \boldsymbol{\Phi}_1(z)\mathbf{K}^{-1}\sqrt{2}\begin{bmatrix} \sqrt{\frac{5}{12}} & \frac{\sqrt{6}}{6} \\ 0 & \frac{1}{2} \end{bmatrix} \tag{9.10}$$

is also a canonical solution of ordered normal form at infinity and is such that $\boldsymbol{\Phi}_0(z) = \overline{\boldsymbol{\Phi}_0(\bar{z})}$. In view of equations (9.8) and (9.10), we can therefore write equation (9.4) as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Phi}_0(z)\tilde{\boldsymbol{\Phi}}_0(-z), \tag{9.11}$$

where

$$\boldsymbol{\Phi}_0(0)\tilde{\boldsymbol{\Phi}}_0(0) = \mathbf{I} \tag{9.12a}$$

and

$$\lim_{|z|\to\infty} z\boldsymbol{\Phi}_0(z) = \sqrt{2}\begin{bmatrix} \sqrt{\frac{5}{12}} & \frac{\sqrt{6}}{6} \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{9.12b}$$

We note that Cercignani[11] has reported a factorization in the spirit of our equation (9.11). We have been unable, however, to justify some of Cercignani's results [11; pp. 84–85] since, for example, upon 'taking' determinants of his equation (311) we find an inconsistency in the number of poles on the two sides of the equality sign. We have found that the extension of scalar results to the case of matrix Riemann problems, in general, does not follow immediately[6].

10. HALF-RANGE ORTHOGONALITY AND NORMALIZATION INTEGRALS

The half-range orthogonality relations developed by Kuščer, McCormick, and Summerfield[18] for the elementary solutions of the one-speed neutron-transport equation have proved to be useful for establishing concisely the solutions to a scalar singular integral equation somewhat analogous to equation (8.1). We should thus like to prove, in a manner similar to that reported by Siewert[24] for an equation of transfer basic to the scattering of polarized light, the following theorem concerning the half-range orthogonality properties of a subset of our developed normal modes.

Theorem 5. *The eigenvectors* $\mathbf{F}_1(\mu)$, $\mathbf{F}_2(\mu)$, $\mathbf{F}_1(\eta,\mu)$, *and* $\mathbf{F}_2(\eta,\mu)$, $\eta\in(0,\infty)$, *are orthogonal to the related set* $\mathbf{G}_1(\mu)$, $\mathbf{G}_2(\mu)$, $\mathbf{G}_1(\eta,\mu)$ *and* $\mathbf{G}_2(\eta,\mu)$, $\eta\in(0,\infty)$, *on the half-range*, $\mu\in(0,\infty)$, *in the sense that*

$$\int_0^\infty \tilde{\mathbf{G}}(\xi',\mu)\mathbf{F}(\xi,\mu)\,\mathrm{e}^{-\mu^2}\mu\,\mathrm{d}\mu = 0,\ \xi\neq\xi';\ \xi,\xi' = \infty \text{ or } \in(0,\infty). \tag{10.1}$$

Here $\mathbf{F}(\xi,\mu)$, $\xi=\infty$ or $\in(0,\infty)$, denotes any of the eigenvectors $\mathbf{F}_1(\mu)$, $\mathbf{F}_2(\mu)$, for $\xi=\infty$, or $\mathbf{F}_1(\eta,\mu)$, $\mathbf{F}_2(\eta,\mu)$, for $\eta\in(0,\infty)$. In a similar manner, $\mathbf{G}(\xi,\mu)$, represents either

$$\mathbf{G}_\alpha(\mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}_1^{-1}\mathbf{Q}^{-1}(\mu)\mathbf{F}_\alpha(\mu),\ \alpha = 1 \text{ or } 2, \tag{10.2a}$$

or

$$\mathbf{G}_\alpha(\eta,\mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{Q}^{-1}(\mu)\mathbf{F}_\alpha(\eta,\mu),\ \eta\in(0,\infty),\ \alpha = 1 \text{ or } 2. \tag{10.2b}$$

In addition

$$\tilde{\mathbf{H}}_1 = \int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\,\mu\,\mathrm{d}\mu \tag{10.3}$$

and $\mathbf{H}(\mu)$ is the $\mathbf{H}$ matrix basic to our half-range analysis. In section 11 we prove the existence of a unique solution to the system of singular integral equations

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P\int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta-\mu},\ \mu\in(0,\infty), \tag{10.4a}$$

plus the constraint

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\,d\mu = \mathbf{I}, \tag{10.4b}$$

which we take to specify $\mathbf{H}(\mu)$. As shall be shown, $\mathbf{H}(\mu)$ can be expressed in terms of $\boldsymbol{\Phi}_0(z)$, our canonical solution, of ordered normal form at infinity, of the matrix Riemann problem defined by equation (9.1); that is

$$\mathbf{H}(\mu) = \tilde{\boldsymbol{\Phi}}_0^{-1}(-\mu)\tilde{\boldsymbol{\Phi}}_0(0), \tag{10.5}$$

which can be extended to the complex plane cut along the negative real axis to yield a factorization of $\boldsymbol{\Lambda}(z)$:

$$\boldsymbol{\Lambda}(z) = \tilde{\mathbf{H}}^{-1}(z)\mathbf{H}^{-1}(-z). \tag{10.6}$$

To establish our Theorem 5, we first pre-multiply equation (4.4) by $\tilde{\mathbf{G}}(\xi',\mu)\,e^{-\mu^2}/\xi$, we then post-multiply the transpose of equation (4.4), having changed ξ to ξ', by $\tilde{\mathbf{Q}}^{-1}(\mu)\tilde{\mathbf{H}}^{-1}(\xi')\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi,\mu)\exp(-\mu^2)/\xi'$, integrate the two resulting equations over μ from 0 to ∞ and subtract the two equations, one from the other, to obtain

$$\left(\frac{1}{\xi}-\frac{1}{\xi'}\right)\int_0^\infty \tilde{\mathbf{G}}(\xi',\mu)\mathbf{F}(\xi,\mu)\,e^{-\mu^2}\mu\,d\mu = \frac{1}{\sqrt{\pi}}[K_1(\xi',\xi)-K_2(\xi',\xi)],\ \xi \text{ and } \xi'>0. \tag{10.7}$$

Here

$$K_1(\xi',\xi) = \tilde{\mathbf{M}}(\xi')\tilde{\mathbf{H}}^{-1}(\xi')\int_0^\infty \tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi,\mu)\,e^{-\mu^2}\,d\mu \tag{10.8}$$

and

$$K_2(\xi',\xi) = \int_0^\infty \tilde{\mathbf{F}}(\xi',\mu)\tilde{\mathbf{Q}}^{-1}(\mu)\tilde{\mathbf{H}}^{-1}(\xi')\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu)\,e^{-\mu^2}\,d\mu\ \mathbf{M}(\xi). \tag{10.9}$$

If now, in the manner similar to that previously reported[24], we make use of equations (4.5), (4.11), (4.13), (4.14) and (10.4) to evaluate equations (10.8) and (10.9) for all appropriate ξ and ξ', we find

$$K_1(\xi',\xi) = K_2(\xi',\xi);\ \xi'\in(0,\infty),\ \xi=\infty \text{ or } \in(0,\infty), \tag{10.10}$$

and from equation (10.7) we obtain

$$\left(\frac{1}{\xi}-\frac{1}{\xi'}\right)\int_0^\infty \tilde{\mathbf{G}}(\xi',\mu)\mathbf{F}(\xi,\mu)\,e^{-\mu^2}\mu\,d\mu = 0;\ \xi,\xi'>0, \tag{10.11}$$

which proves the theorem. We have only established equation (10.11) formally for $\xi'\neq\infty$. However, considering that case separately, we do find that $\mathbf{G}_\alpha(\mu)$, $\alpha = 1$ or 2, is orthogonal to $\mathrm{F}(\xi,\mu)$ in the sense of Theorem 5. Of course, since $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ both correspond to the eigenvalue $\xi=\infty$, equation (10.1) does not ensure that the inner product, in the sense of Theorem 5, of $\mathbf{G}_1(\mu)$ with $\mathbf{F}_2(\mu)$ and $\mathbf{G}_2(\mu)$ with $\mathbf{F}_1(\mu)$ is zero.

However, for this special case we have carried out the algebra prescribed by equation (10.1) to show explicitly that

$$\int_0^\infty \tilde{\mathbf{G}}_\alpha(\mu)\mathbf{F}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0,\ \alpha \neq \beta, \tag{10.12}$$

so that all of the half-range eigenvectors are orthogonal in the manner of Theorem 5.

Having established the required half-range orthogonality results, we should now like to consider again the normalized solutions given by equation (6.7) in order to present our half-range normalization integrals in a form analogous to that used for the full-range theory. We consider then the half-range adjoint set

$$\boldsymbol{\Theta}_\alpha(\mu) = \frac{1}{\sqrt{\pi}}\mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}_1^{-1}\mathbf{Q}^{-1}(\mu)\boldsymbol{\Phi}_\alpha(\mu),\ \alpha = 1 \text{ and } 2, \tag{10.13a}$$

and

$$\boldsymbol{\Theta}_\alpha(\eta,\mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{Q}^{-1}(\mu)\mathbf{X}_\alpha(\eta,\mu),\ \eta \in (0,\infty),\ \alpha = 1 \text{ and } 2, \tag{10.13b}$$

where the vectors $\mathbf{X}_\alpha(\eta,\mu)$, $\alpha = 1$ and 2, are given by equations (6.11e) and (6.11f); we can therefore summarize our results in the manner

$$\int_0^\infty \tilde{\boldsymbol{\Theta}}_\alpha(\mu)\boldsymbol{\Phi}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = \delta_{\alpha,\beta};\ \alpha,\beta = 1 \text{ or } 2, \tag{10.14a}$$

$$\int_0^\infty \tilde{\boldsymbol{\Theta}}_\alpha(\mu)\boldsymbol{\Phi}_\beta(\eta,\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0,\ \eta \in (0,\infty);\ \alpha,\beta = 1 \text{ or } 2, \tag{10.14b}$$

$$\int_0^\infty \tilde{\boldsymbol{\Theta}}_\alpha(\eta',\mu)\boldsymbol{\Phi}_\beta(\eta,\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = \delta_{\alpha,\beta}\delta(\eta-\eta');\ \eta,\eta' \in (0,\infty),\ \alpha,\beta = 1 \text{ or } 2, \tag{10.14c}$$

and

$$\int_0^\infty \tilde{\boldsymbol{\Theta}}_\alpha(\eta,\mu)\boldsymbol{\Phi}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0;\ \eta \in (0,\infty),\ \alpha,\beta = 1 \text{ or } 2. \tag{10.14d}$$

With the half-range formalism thus established, we note that the solutions for all expansion coefficients in equations of the form

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=1}^{2}\int_0^\infty A_\alpha(\eta)\mathbf{F}_\alpha(\eta,\mu)\,\mathrm{d}\eta,\ \mu \in (0,\infty), \tag{10.15}$$

can be expressed concisely as

$$A_\alpha = \int_0^\infty \tilde{\boldsymbol{\Theta}}_\alpha(\mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu,\ \alpha = 1 \text{ and } 2, \tag{10.16a}$$

and

$$A_\alpha(\eta) = \int_0^\infty \tilde{\boldsymbol{\Theta}}_\alpha(\eta,\mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu,\ \alpha = 1 \text{ and } 2. \tag{10.16b}$$

We note that a set of integrals of the form

$$T(\xi', \xi) = \int_0^\infty \tilde{\mathbf{\Theta}}(\xi', \mu)\boldsymbol{\Phi}(-\xi, \mu)\, \mathrm{e}^{-\mu^2}\, \mu\, \mathrm{d}\mu,\ \xi \text{ and } \xi' > 0. \tag{10.17}$$

has been evaluated and is listed elsewhere[16].

11. EXISTENCE AND UNIQUENESS OF THE H MATRIX

We first wish to prove

Theorem 6. *The equations*

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P \int_0^\infty \tilde{\mathbf{H}}(\eta)\mathbf{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta - \mu},\ \mu \in (0, \infty), \tag{11.1a}$$

and

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\, \mathrm{d}\mu = \mathbf{I} \tag{11.1b}$$

possess a unique solution in the class of functions continuous on every open interval of the positive real axis.

Though for the sake of brevity we do not give an explicit derivation of equations (11.1), we note that the **H** matrix specified by equations (11.1) is sufficient for establishing the half-range orthogonality theorem.

To prove Theorem VI we make use of the equivalence of the given singular-integral equations to a certain matrix version of the Riemann problem. In the manner of Muskhelishvili[21], we introduce the matrix

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_0^\infty \tilde{\mathbf{H}}(\eta)\mathbf{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta - z} \tag{11.2}$$

which is analytic in the plane cut along the real axis and vanishes at least as fast as $1/z$ as $|z|$ tends to infinity. The Plemelj formulae[21] can be used with equation (11.2) to yield

$$\pi i[\mathbf{N}^+(\mu) + \mathbf{N}^-(\mu)] = P\int_0^\infty \tilde{\mathbf{H}}(\eta)\mathbf{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta - \mu},\ \mu \in (0, \infty), \tag{11.3}$$

and

$$\mathbf{N}^+(\mu) - \mathbf{N}^-(\mu) = \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu),\ \mu \in (0, \infty). \tag{11.4}$$

If we make use of equations (5.10), (11.3) and (11.4) then equation (11.1a) can be cast in the equivalent form of an inhomogeneous Riemann problem:

$$\tilde{\mathbf{N}}^+(\mu) = \mathbf{G}(\mu)\tilde{\mathbf{N}}^-(\mu) + \mathbf{\Psi}(\mu)[\mathbf{\Lambda}^-(\mu)]^{-1},\ \mu \in (0, \infty), \tag{11.5}$$

where

$$\mathbf{G}(\mu) = \mathbf{\Lambda}^+(\mu)[\mathbf{\Lambda}^-(\mu)]^{-1}. \tag{11.6}$$

The solution to equation (11.5) can be written as

$$\tilde{\mathbf{N}}(z) = \frac{1}{2\pi i}\boldsymbol{\Phi}(z)\left[\int_0^\infty \mathbf{K}(\eta)\frac{\mathrm{d}\eta}{\eta - z} + \hat{\mathbf{P}}(z)\right] \tag{11.7}$$

where $\hat{\mathbf{P}}(z)$ is a matrix of polynomials,

$$\mathbf{K}(\mu) = [\boldsymbol{\Phi}^+(\mu)]^{-1}\boldsymbol{\Psi}(\mu)[\boldsymbol{\Lambda}^-(\mu)]^{-1}, \tag{11.8}$$

and $\boldsymbol{\Phi}(z)$ is any canonical solution (of ordered normal form at infinity) to the homogeneous Riemann problem defined by equation (9.1). In order that equation (11.7) have behavior as $|z|$ approaches infinity consistent with equation (11.2), we must take $\hat{\mathbf{P}}(z)$ to be a constant matrix.

Following closely Siewert and Burniston's[25] work on the scattering of polarized light, we can now use the constraint to specify uniquely the constant $\hat{\mathbf{P}}(z)$ in equation (11.7). It thus follows that equation (11.7) along with equations (9.11) and (11.4) yields the result

$$\mathbf{H}(\mu) = \tilde{\boldsymbol{\Phi}}_0^{-1}(-\mu)\tilde{\boldsymbol{\Phi}}_0(0),\ \mu \in (0, \infty), \tag{11.9}$$

where $\boldsymbol{\Phi}_0(z)$ is the canonical solution (of ordered normal form at infinity) used in equation (9.11). We note from equation (11.9), since $\boldsymbol{\Phi}_0(z) = \overline{\boldsymbol{\Phi}_0(\bar{z})}$, that $\mathbf{H}(\mu)$ is real for $\mu \in (0, \infty)$. Further, equation (11.9) can be used to extend the definition of $\mathbf{H}(\mu)$ to the complex plane:

$$\mathbf{H}(z) = \tilde{\boldsymbol{\Phi}}_0^{-1}(-z)\tilde{\boldsymbol{\Phi}}_0(0), \tag{11.10}$$

so that the $\boldsymbol{\Lambda}$ matrix can now be factored as

$$\boldsymbol{\Lambda}(z) = \tilde{\mathbf{H}}^{-1}(-z)\mathbf{H}^{-1}(z).$$

We note that equations (5.10), (9.11) and (11.10) can be used in the Cauchy integral representation

$$\boldsymbol{\Phi}_0(z) = \frac{1}{2\pi i}\int_0^\infty [\boldsymbol{\Phi}_0^+(\mu) - \boldsymbol{\Phi}_0^-(\mu)]\frac{\mathrm{d}\mu}{\mu - z} \tag{11.11}$$

to obtain

$$\mathbf{H}(z) = \mathbf{I} + z\mathbf{H}(z)\int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + z} \tag{11.12}$$

or

$$\mathbf{H}(\mu) = \mathbf{I} + \mu\mathbf{H}(\mu)\int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + \mu},\ \mu \in [0, \infty). \tag{11.13}$$

Since we have established the existence of a unique solution to equations (11.1) and developed equation (11.13) specifically to be used, along with equation (11.1b), as the basis for our procedure for computing the **H** matrix, it follows that we require proof of

Theorem 7. *The equations*

$$\mathbf{H}(\mu) = \mathbf{I} + \mu \mathbf{H}(\mu) \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + \mu}, \ \mu \in [0, \infty), \tag{11.14a}$$

and

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\,\mathrm{d}\mu = \mathbf{I} \tag{11.14b}$$

possess a unique solution in the class of functions continuous on every open interval of the positive real axis.

We have shown that equations (11.1) possess a unique solution; thus we need only show that any solution of equation (11.14a) is also a solution of equation (11.1a). We first write equation (11.14a) as

$$\mathbf{H}(\mu)\left[\mathbf{I} - \mu \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + \mu}\right] = \mathbf{I} \tag{11.15a}$$

or, alternatively,

$$\left[\mathbf{I} - \mu \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + \mu}\right]\mathbf{H}(\mu) = \mathbf{I}. \tag{11.15b}$$

If the transpose of equation (11.15b) is post-multiplied by

$$\mathbf{I} + \mu P \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta - \mu}$$

then, after making use of some partial-fraction analysis and equations (4.15) and (11.15), we obtain

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta - \mu}, \ \mu \in (0, \infty), \tag{11.16}$$

which proves Theorem 7.

12. AN EXPEDIENT METHOD FOR COMPUTING THE H MATRIX

It is clear from the previous sections of this paper that the **H** matrix is the basic quantity required in the solutions of problems defined in terms of equation (4.18) and specified by half-range, $\mu \in (0, \infty)$, boundary conditions. It is also apparent from the analysis of section 11 that the basic proofs regarding the existence and uniqueness of the **H** matrix have been established; however, to demonstrate the utility of our analysis, we must now establish a procedure by which we can compute the **H** matrix accurately and efficiently.

As we have discussed, the **H** matrix is uniquely specified by the nonlinear equation

$$\mathbf{H}(\mu)=\mathbf{I}+\mu\mathbf{H}(\mu)\int_0^\infty \tilde{\mathbf{H}}(\eta)\mathbf{\Psi}(\eta)\frac{d\eta}{\eta+\mu},\ \mu\in[0,\infty), \tag{12.1a}$$

and the constraint

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\,d\mu=\mathbf{I}. \tag{12.1b}$$

Rather than seek a numerical solution to equation (12.1a) which must also satisfy equation (12.1b), we prefer [15, 22] first to write

$$\mathbf{H}(\mu)=(1+\mu)\mathbf{L}(\mu),\ \mu\in[0,\infty). \tag{12.2}$$

If we now substitute equation (12.2) into equation (12.1a), perform some elementary partial-fraction analysis, and make use of the constraint, equation (12.1b), then we find that $\mathbf{L}(\mu)$ must satisfy

$$\mathbf{L}(\mu)=\mathbf{I}+\mu\mathbf{L}(\mu)\int_0^\infty (1-\eta^2)\tilde{\mathbf{L}}(\eta)\mathbf{\Psi}(\eta)\frac{d\eta}{\eta+\mu},\ \mu\in[0,\infty), \tag{12.3a}$$

and

$$\int_0^\infty \tilde{\mathbf{L}}(\mu)\mathbf{\Psi}(\mu)(1+\mu)\,d\mu=\mathbf{I}. \tag{12.3b}$$

It follows from Theorem 7 that equations (12.3) have a unique solution. We regard equations (12.3) as the basic equations to be solved numerically because an iterative procedure based on these equations has proven to converge faster than a similar iterative solution of equations (12.1).

For calculational convenience, we now prefer to make in equations (12.3) the change of variable

$$t=\frac{\mu}{1+\mu} \tag{12.4}$$

and thus to write

$$\mathbf{L}_*(t)=\mathbf{I}+t\mathbf{L}_*(t)\int_0^1 \frac{1-2s}{(1-s)^3}\tilde{\mathbf{L}}_*(s)\mathbf{\Psi}_*(s)\frac{ds}{t(1-s)+s(1-t)},\ t\in[0,1), \tag{12.5}$$

where we have introduced the notation $g_*(t)=g[t/(1-t)]$. We have found that equation (12.5) can be solved quite effectively by iteration.

The computations were performed in double-precision arithmetic on an IBM 370/165 computer, and we used an improved Gaussian-quadrature [17] representation of the integration process. The iterative procedure was terminated when successive calculations of $\mathbf{L}_*(t)$ differed by no more than 10^{-15}.

To substantiate confidence in our computations, several "checks" were incorporated in the calculation. As expected $\mathbf{L}_*(t)$ satisfied

$$\mathbf{I}-\int_0^1 \tilde{\mathbf{L}}_*(t)\mathbf{\Psi}_*(t)\frac{dt}{(1-t)^3}=\mathbf{0}, \tag{12.6}$$

an identity corresponding to the constraint, equation (12.1b), to thirteen significant figures. The equation in terms of $\mathbf{L}_*(t)$ corresponding to the identity[16]

$$2\int_0^\infty \mathbf{\Psi}(\mu)\,\mu^2 d\mu - \int_0^1 \mathbf{\Psi}(\mu)\mathbf{H}(\mu)\,\mu\,d\mu \int_0^1 \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\,\mu\,d\mu = \mathbf{0} \tag{12.7}$$

was also verified to thirteen significant figures.

The analytical solution[16]

$$H(z) = \det \mathbf{H}(z) = \sqrt{\frac{12}{5}}z^2 \exp\left\{-\frac{1}{\pi}\int_0^\infty \arg \Lambda^+(\mu)\frac{d\mu}{\mu + z}\right\}, \tag{12.8}$$

where $\arg \Lambda^+(0) = -2\pi$, can be shown to satisfy

$$H(\mu) = 1 + \mu H(\mu)\int_0^\infty f(\eta)H(\eta)\frac{d\eta}{\eta + \mu},\ \mu \in [0, \infty), \tag{12.9}$$

where

$$f(\eta) = \frac{1}{3}\frac{1}{\sqrt{\pi}}e^{-\eta^2}\left[\frac{11}{2} - \eta^2 - 8\eta\, e^{-\eta^2}\int_0^\eta e^{t^2}dt\right]. \tag{12.10}$$

Rather than solve equation (12.9) and the appropriate constraint for $H(\mu)$ we prefer to write

$$H(\mu) = (1 + \mu)^2 L(\mu),\ \mu \in [0, \infty). \tag{12.11}$$

If we substitute equation (12.11) into equation (12.9), perform some partial-fraction analysis and make use of two identities[16] for $H(\mu)$ then we find that (after an appropriate change of variables) $L_*(t) = \det \mathbf{L}_*(t)$ must satisfy

$$\frac{1}{L_*(t)} = 1 - t\int_0^1 \frac{(1-2s)^2}{(1-s)^5} f_*(s)L_*(s)\frac{ds}{s(1-t) + t(1-s)},\ t \in [0, 1). \tag{12.12}$$

We have compared $L_*(t)$ as computed from equation (12.5) to a direct solution of equation (12.12) and the appropriate constraint to find agreement to nine significant figures.

Finally the number of quadrature points used to represent the integration process was increased to suggest that the numerical values of the $\mathbf{L}_*$ matrix given in the accompanying Table (1) were insensitive to further refinements in the quadrature scheme.

13. AN APPLICATION OF THE THEORY: THE TEMPERATURE SLIP PROBLEM

We consider the effect of a body surface on the behavior of the particle distribution function of a rarefied gaseous medium. It is known that, in the absence of boundaries, the particle distribution function in a gas with slowly varying physical parameters obeys the Chapman–Enskog equations (and therefore the macroscopic variables obey the

Table 1. The $\mathbf{L}_+$ matrix

t	$L_{+11}(t)$	$L_{+12}(t)$	$L_{+21}(t)$	$L_{+22}(t)$
0·0	1·0	0·0	0·0	1·0
0·05	1·05781	−0·0409718	−0·0406202	1·08988
0·10	1·08842	−0·0702992	−0·0691920	1·14897
0·15	1·10976	−0·0960791	−0·0939297	1·19824
0·20	1·12527	−0·119749	−0·116314	1·24158
0·25	1·13648	−0·141931	−0·136989	1·28073
0·30	1·14424	−0·162963	−0·156307	1·31664
0·35	1·14909	−0·183053	−0·174483	1·34992
0·40	1·15139	−0·202341	−0·191658	1·38097
0·45	1·15138	−0·220921	−0·207929	1·41009
0·50	1·14926	−0·238862	−0·223361	1·43748
0·55	1·14514	−0·256214	−0·238000	1·46333
0·60	1·13912	−0·273011	−0·251873	1·48774
0·65	1·13125	−0·298278	−0·264997	1·51082
0·70	1·12157	−0·305030	−0·277373	1·53265
0·75	1·11008	−0·320274	−0·288995	1·55328
0·80	1·09678	−0·335010	−0·299844	1·57276
0·85	1·08162	−0·349228	−0·309887	1·59111
0·90	1·06454	−0·362913	−0·319081	1·60835
0·95	1·04545	−0·376036	−0·327361	1·62449
0·99	1·02864	−0·386105	−0·333271	1·63660

Navier–Stokes equations). Near the body surface, the behavior of the gas is described by a rarefied Knudsen layer in which the collisional effects are only of secondary importance. It is natural to ask how the outer Chapman–Enskog (or Navier–Stokes) region can be matched consistently with the inner Knudsen layer. Saying it differently, we ask what are the velocity and temperature slip boundary conditions at a body surface for the Navier–Stokes equations due to the presence of the Knudsen layer adjacent to the body surface.

To understand the asymptotic behavior of the Knudsen layer, we may stretch locally the coordinate normal to the body surface such that the gas-kinetic motion in the Knudsen layer reduces to a locally defined half-space problem and the kinetic equation takes on a one-dimensional character in the form studied in this paper.

Since the asymptotic boundary condition of the Knudsen layer is given by the Chapman–Enskog equations, the asymptotic form of the particle distribution function is nearly Maxwellian. If we also assume that the effect of the body surface is to re-emit molecules described by a suitably chosen Maxwellian distribution and that the macroscopic variables do not vary appreciably throughout the Knudsen layer, a linearization scheme for the one-dimensional kinetic equation in the sense described in sections 2 and 3 is justified. Based on the constant collision frequency BGK model, the pertinent linearized kinetic equation for the Knudsen layer is that given by equation (3.7). The velocity-slip (or Kramers problem) for this equation for a diffusely reflecting wall has been solved exactly by Cercignani[9] and an accurate velocity-slip coefficient has been calculated,[1]. Although approximate analyses of the associated temperature slip problem have been reported by a number of authors[2, 19, 23, 28, 29], an accurate calculation of the temperature-slip coefficient has not been previously reported. Since the temperature-density effects for the problem for a diffusely reflecting wall are uncoupled from the transverse momentum effects, we will show that an accurate determination of the temperature slip coefficient for steady gas-kinetic motion may be effected by using the method of elementary solutions and the half-range expansion theorem developed in this paper for the vector integrodifferential equation (4.1).

It is straightforward to demonstrate that the heat flux in the normal direction in the Knudsen layer is a constant and since the Chapman–Enskog solution relates the heat flux linearly to the temperature gradient, to match the Chapman–Enskog region and the Knudsen layer it is only necessary to consider an asymptotic boundary condition with a constant temperature gradient. It is interesting to note that such an asymptotic boundary condition, as far as the temperature-density effects are concerned, can be satisfied by taking the asymptotic perturbation distribution function to be

$$h_{asy}(x,\mathbf{c}) = \begin{bmatrix} 1 \\ c_2^2 + c_3^2 - 1 \end{bmatrix}^T \left[\sum_{\alpha=1}^{2} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\beta=3}^{4} A_\beta \boldsymbol{\Psi}_\beta(x,\mu) \right], \tag{13.1}$$

where the $\boldsymbol{\Phi}_\alpha'$s and $\boldsymbol{\Psi}_\beta'$s are the discrete solutions to equation (4.1), and c_2, c_3 are the components of the dimensionless particle velocity in the transverse directions. Since the Chapman–Enskog theory requires the medium to obey the perfect gas law and the pressure in the Knudsen layer far from the wall is a specified constant, we deduce from the definitions

$$n_{asy}(x) \triangleq \pi^{-3/2} \int h_{asy}(x,\mathbf{c})\, e^{-c^2}\, d^3c, \tag{13.2}$$

and

$$T_{asy}(x) \triangleq \tfrac{2}{3}\pi^{-3/2} \int h_{asy}(x,\mathbf{c})(c^2 - \tfrac{3}{2})\, e^{-c^2}\, d^3c, \tag{13.3}$$

the requirements

$$A_2 = -\sqrt{\tfrac{2}{3}}A_1 \quad \text{and} \quad A_4 = -\sqrt{\tfrac{2}{3}}A_3. \tag{13.4}$$

For an asymptotic temperature gradient of unity, it is clear that the temperature slip coefficient ξ defined by $T_{asy}(0) = \xi(d/dx)T_{asy}(x)|_{x=0}$ will be given by $\xi = \epsilon' l_t$ [19], where

$$\epsilon' = \sqrt{\tfrac{3}{2}}A_1, \tag{13.5}$$

$l_t = (4/5)(\mathcal{K}/nk)(m/2kT)^{1/2}$ is a mean free path, and $\mathcal{K}$ is the thermal conductivity.

For a diffusely reflecting wall, the boundary condition at $x = 0$ requires

$$\begin{bmatrix} B \\ 0 \end{bmatrix} = \sum_{\alpha=1}^{4} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_0^\infty A_\alpha(\eta)\boldsymbol{\Phi}_\alpha(\eta,\mu)\, d\eta, \ \mu \in (0,\infty). \tag{13.6}$$

The unknown constant B is related to the density of the gas near the wall and need not be specified for temperature-slip coefficient calculations. We make use of equation (13.4) and the specified asymptotic temperature gradient to write equation (13.6) as

$$\sqrt{\tfrac{3}{2}}\boldsymbol{\Phi}_3(\mu) - \boldsymbol{\Phi}_4(\mu) = A_1\boldsymbol{\Phi}_1(\mu) + (A_2 - B)\boldsymbol{\Phi}_2(\mu) + \sum_{\alpha=1}^{2} \int_0^\infty A_\alpha(\eta)\boldsymbol{\Phi}_\alpha(\eta,\mu)\, d\eta. \tag{13.7}$$

Theorem 3 and equation (10.16a) enable us to solve equation (13.7) for A_1:

$$A_1 = \int_0^\infty \tilde{\Theta}_1(\mu)[\sqrt{\tfrac{3}{2}}\boldsymbol{\Phi}_3(\mu) - \boldsymbol{\Phi}_4(\mu)]\, e^{-\mu^2}\mu\, d\mu. \tag{13.8}$$

It is clear from the definition of $\Theta_1(\mu)$ that A_1 may be expressed in terms of appropriate moments of the **H** matrix discussed in section 11. The numerical procedure used to evaluate integrals involving the **H** matrix was given in section 12. We find

$$\epsilon' = \tfrac{5}{8}\pi^{1/2}(1{\cdot}17597). \tag{13.9}$$

This compares with the variational result of $\epsilon' = \frac{5}{8}\pi^{1/2}$ (1·1621), [2, 19, 23], Wang Chang and Uhlenbeck's result of $\epsilon' = \frac{5}{8}\pi^{1/2}$ (1·150), [28], and Welander's value of $\epsilon' = \frac{5}{8}\pi^{1/2}$ (1·173), [29]. We believe our result to be accurate to the number of significant figures quoted.

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Résumé—La méthode des solutions élémentaires est employée pour résoudre deux équations integro-différentielles couplées pour la détermination des effets du rapport température-densité, dans un modèle BGK linéarisé dans la théorie cinétique des gaz. L'état complet de tout le domaine et les théorèmes d'orthogonalité sont démontrés pour les modes normaux développés et la fonction de Green pour un milieu infini est établie comme illustration du formalisme sur tout le domaine.

Le problème de Riemen approprié à une matrice homogène est étudiée et l'état complet de tout le domaine et les théorèmes d'orthogonalité sont démontrés pour un certain sous-système des modes normaux. Les théorèmes nécessaires d'existence et d'unicité concernant la matrice H, fondamentale pour l'analyse du domaine complet, sont démontrés et une méthode de calcul précise et efficace est présentée. Le problème du glissement de température d'un demiespace est résolu analytiquement et une valeur très précise du coefficient de glissement de température est donnée.

Zusammenfassung—Das Verfahren elementarer Lösungen wird verwendet, um zwei gekuppelte Integrodifferentialgleichungen zu lösen, die genügen, um die Temperatur-Dichtewirkungen in einem linearisierten BGK-Modell in der kinetischen Theorie von Gasen zu bestimmen.

Vollbereichs-Vollständigkeits- und Orthogonalitätstheoreme werden für die entwickelten Normalformen bewiesen und Green's Funktion des unendlichen Stoffes wird als eine Illustration des Vollbereichsformalismus konstruiert.

Das entsprechende Riemann'sche Problem einer homogenen Matrix wird besprochen und Halbbereichs-Vollständigkeits- und Orthogonalitätstheoreme werden für eine bestimmte Untergruppe der Normalformen bewiesen. Die für die $\underset{\sim}{H}$-Matrix belangreichen erforderlichen Existenz- und Eindeutigkeitstheoreme, grundlegend für die Halbbereichs-Analyse, werden bewiesen, und eine genaue und wirksame Berechnungsmethode wird besprochen. Das Temperaturgleitproblem des Halbraumes wird analytisch gelöst und ein sehr genauer Wert des Temperaturgleitkoeffizienten wird berichtet.

Sommario—Il metodo delle soluzioni elementari viene usato per risolvere due equazioni integrodifferenziali accoppiate, sufficienti a determinare gli effetti della densità di temperatura in un modello BGK linearizzato nella teoria cinetica dei gas.

Vengono dimostrati i teoremi della completezza e dell'ortogonalità su tutta la gamma per i modi normali sviluppati e, per illustrare il formalismo sull'intera gamma, viene costruita la funzione di Green per un mezzo infinito.

Viene discusso il problema di Riemann per una matrice omogenea appropriata e, per un certo sottoassieme dei modi normali, vengono dimostrati i problemi di completezza e ortogonalità per la semigamma. Vengono dimostrati i necessari teoremi di esistenza e unicità relativi alla matrice $\underset{\sim}{H}$, fondamentali per l'analisi della semigamma, e viene discusso un metodo di calcolo accurato ed efficiente. Il problema temperatura/scorrimento del semispazio viene risolto analiticamente e viene dato un valore molto accurato del coefficiente temperatura/scorrimento.

Абстракт — Применен метод элементарных решений для решения двух интегро-дифференциальных уравнений достаточных для определения эффектов температуры и плотности в линеаризованной модели (BGK) в кинетической теории газов. Даны доказательства теорем об общей полноте и ортогональности для развиваемых нормальных режимов, построена функция Грина для бесконечной среды в качестве иллюстрации общего формализма. Обсуждена соответствующая римановская проблема однородной матрицы, даны доказательства теорем о половинной полноте и ортогональности для определенной подсистемы нормальных режимов. Доказаны искомые теоремы о существовании и единственности относительно матрицы H, лежащие в основе анализа половинной полноты, а также обсужден точный рациональный метод вычисления. Аналитически решена проблема полупространственного скольжения и температуры, сообщено высокоточное значение коэффициента скольжения и температуры.