JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 46, 329–337 (1974)

Solutions of the Equation $ze^z = a(z + b)$

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The theory of complex variables is used to develop exact closed-form solutions for the, in general, complex zeros of the exponential polynomial $F(z) = z \exp z - a(z + b)$, a complex and b real. The established zeros are related to canonical solutions of suitably posed Riemann problems and are expressed ultimately in terms of elementary quadratures.

I. INTRODUCTION

In a recent paper [1] we reported exact analytical solutions of the transcendental equation

$$ze^z = a, \qquad a \text{ complex}, \tag{1}$$

which is basic to the analysis of a class of differential-difference equations [2]. Here the same method [3] will be used to solve the slightly more general equation

$$(l + m\zeta)\, e^\zeta = p + q\zeta. \tag{2}$$

We immediately note that if $mq = 0$ then Eq. (2) may be reduced to Eq. (1), while if $m \neq 0$ and $mp = ql$ then Eq. (2) is simply

$$me^\zeta = q. \tag{3}$$

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Assuming that neither of these two elementary cases apply, we can transform Eq. (2) to the equivalent form

$$F(z) \equiv ze^z - a(z + b) = 0. \tag{4}$$

We note that Wright [4] has considered Eq. (2), for real values of the parameters; he has also enumerated the solutions and has given some bounds for the various solutions. In addition to an exhaustive bibliography on this problem, Wright's paper [4] reports an algorithm that can be used for computational purposes. Here we allow a to be complex and subsequently develop exact analytical expressions for all of the zeros of $F(z)$. It will be clear from the ensuing analysis that b also may be complex, but we prefer to report only the more concise formalism resulting from considering b to be real.

II. Analysis

It is apparent that the zeros z_k of $F(z)$, as given by Eq. (4), are the zeros of the functions

$$\Lambda_k(z) = \alpha + \log(z + b) - z - \log z + 2k\pi i, \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{5}$$

where

$$\alpha = \log a, \qquad a \neq 0, \tag{6}$$

and here $\log z$ denotes the principal branch of the log function. We observe therefore that $\Lambda_k(z)$ is analytic in the complex plane cut along the real axis from $-b$ to 0 for $b > 0$, or cut from 0 to $-b$ for $b < 0$.

We shall investigate the two cases separately:

(1) $b > 0 \Rightarrow \Lambda_k(z)$ is analytic in the plane cut on $[-b, 0]$

(2) $b < 0 \Rightarrow \Lambda_k(z)$ is analytic in the plane cut on $[0, -b]$.

For case (1), $b > 0$, we can write the limiting values of $\Lambda_k(z)$ as z approaches the cut from above (+) and below (−) as

$$\Lambda_k^{\pm}(t) = \alpha + \ln(t + b) - t - \ln|t| + (2k \mp 1)\pi i, \qquad t \in (-b, 0), \ b > 0, \tag{7}$$

Note that neither $\Lambda_k^{+}(t)$ nor $\Lambda_k^{-}(t)$ can vanish on the cut except for the two special cases

(i) $a \in (-\infty, 0)$ and $k = 0$

(ii) $a \in (-\infty, 0)$ and $k = -1$.

For these two special cases, we see that

$$\Lambda_0^+(t) = \Lambda_{-1}^-(t), \qquad a \in (-\infty, 0), \tag{8}$$

and hence we shall require the zeros of

$$\omega(t) = \ln |a| + \ln(t + b) - t - \ln |t|, \qquad t \in (-b, 0), \quad a \in (-\infty, 0). \tag{9}$$

Elementary considerations are sufficient to show that $\omega(t)$ has at least one zero, say t_0, for $t \in (-b, 0)$. We shall discuss the two special cases (i) and (ii) separately and thus first consider all values of a, b, and k such that

$$\{a, b, k\} \in D \Rightarrow k = 0, \pm 1, \pm 2, \pm 3, \ldots, \qquad \text{if} \quad a \notin (-\infty, 0] \quad \text{and} \quad b > 0,$$

or

$$\{a, b, k\} \in D \Rightarrow k = 1, \pm 2, \pm 3, \pm 4, \ldots, \qquad \text{if} \quad a \in (-\infty, 0) \quad \text{and} \quad b > 0.$$

The argument principle [5] can now be used to show, for $\{a, b, k\} \in D$, that $\Lambda_0(z)$ has precisely two zeros, say z_{01} and z_{02}, in the plane cut along $[-b, 0]$ and that $\Lambda_k(z)$, $k = \pm 1, \pm 2, \pm 3, \ldots$, has only one zero z_k in the same cut plane. It therefore follows that

$$F_0(z) = -\frac{\Lambda_0(z)}{(z - z_{01})(z - z_{02})}, \qquad a \notin (-\infty, 0], \quad b > 0, \tag{10}$$

is analytic and nonvanishing in the same cut plane, and thus $F_0(z)$ is a canonical solution of the Riemann problem [6, 7]

$$F_0^+(t) = G_0(t) F_0^-(t), \qquad t \in (-b, 0), \tag{11}$$

where the Riemann coefficient is

$$G_0(t) = \frac{\Lambda_0^+(t)}{\Lambda_0^-(t)}. \tag{12}$$

If we define $G_0(-b) = G_0(0) = 1$, then it follows that $G_0(t)$ is Hölder continuous on $(-b, 0)$, but fails to be Hölder at the two endpoints. We conclude that the (suitably normalized) canonical solution of Eq. (11) can be written as

$$F_0(z) = \frac{1}{z + b} \exp\left[\frac{1}{2\pi i} \int_{-b}^{0} \log G_0(t) \frac{dt}{t - z}\right], \qquad a \notin (-\infty, 0], \quad b > 0, \tag{13}$$

where $\log G_0(t)$ is continuous for $t \in (-b, 0)$ and such that

$$\log G_0(-b) = -2\pi i.$$

Equation (13) can now be entered into Eq. (10) to yield

$$(z_{01} - z)(z_{02} - z) = -(z + b)\Lambda_0(z) \exp\left[\frac{1}{2\pi i}\int_0^b \log G_0(-\tau)\frac{d\tau}{\tau + z}\right]. \tag{14}$$

If we now evaluate Eq. (14) at two convenient points, say $z = \gamma$ and $z = \eta$, off the cut then the resulting two equations can be solved simultaneously to yield

$$z_{01} = -B(\gamma, \eta) - (B^2(\gamma, \eta) - C(\gamma, \eta))^{1/2}, \qquad a \notin (-\infty, 0], \quad b > 0, \tag{15a}$$

and

$$z_{02} = -B(\gamma, \eta) + (B^2(\gamma, \eta) - C(\gamma, \eta))^{1/2}, \qquad a \notin (-\infty, 0], \quad b > 0, \tag{15b}$$

where

$$B(\gamma, \eta) = \frac{1}{2}\left[\frac{K(\gamma) - K(\eta) - \gamma^2 + \eta^2}{\gamma - \eta}\right] \tag{16}$$

and

$$C(\gamma, \eta) = \left[\frac{\gamma K(\eta) - \eta K(\gamma) + \gamma\eta(\gamma - \eta)}{\gamma - \eta}\right]. \tag{17}$$

In addition we have introduced

$$K(z) = -(z + b)\Lambda_0(z) \exp\left[\frac{1}{2\pi i}\int_0^b \log G_0(-\tau)\frac{d\tau}{\tau + z}\right]. \tag{18}$$

Note that the closed-form solutions given by Eqs. (15) contain two free parameters γ and η, the choice of which can yield various equivalent forms and can be used to advantage for computational purposes. In a similar manner, it is apparent that

$$F_k(z) = \frac{\Lambda_k(z)}{z_k - z}, \qquad \{a, b, k\} \in D, \quad k \neq 0, \tag{19}$$

is analytic and nonvanishing in the cut plane, and therefore $F_k(z)$ is a canonical solution of the Riemann problem

$$F_k^+(t) = G_k(t) F_k^-(t), \qquad t \in (-b, 0), \tag{20}$$

where

$$G_k(t) = \frac{\Lambda_k^+(t)}{\Lambda_k^-(t)}. \tag{21}$$

As before, we define $G_k(-b) = G_k(0) = 1$ and thus can write

$$F_k(z) = \exp\left[\frac{1}{2\pi i}\int_{-b}^{0} \log G_k(t)\frac{dt}{t-z}\right], \qquad \{a, b, k\} \in D, \quad k \neq 0, \tag{22}$$

where $\log G_k(t)$ is continuous for $t \in (-b, 0)$ and such that

$$\log G_k(-b) = \log G_k(0) = 0.$$

With $F_k(z)$ established, we can now solve Eq. (19) to obtain the explicit closed-form result

$$z_k = z + [\alpha + \log(z+b) - z - \log z + 2k\pi i]$$
$$\times \exp\left[\frac{1}{2\pi i}\int_{0}^{b} \log G_k(-\tau)\frac{d\tau}{\tau+z}\right], \qquad \{a, b, k\} \in D, \quad k \neq 0. \tag{23}$$

To complete the analysis for $b > 0$, we must now consider the two special cases (i) and (ii), and thus we wish to introduce

$$\Omega(z) = \ln|a| + \log(z+b) - z - \log z + \pi i, \qquad a \in (-\infty, 0), \quad b > 0, \tag{24}$$

where by $\log z$ we now denote that branch of the log-function in the plane cut along the *positive* real axis, such that $0 < \arg z < 2\pi$. With the log function so defined, it follows that $\Omega(z)$ is analytic in the plane cut along $(-\infty, -b]\ U[0, \infty)$ and such that

$$\Omega(t) = \omega(t), \qquad t \in (-b, 0), \quad b > 0. \tag{25}$$

The argument principle can now be used to show that $\Omega(z)$ has three zeros in the cut plane, and since $\Omega(z) = \Lambda_0(z)$ for $y > 0$, $\Omega(z) = \Lambda_{-1}(z)$ for $y < 0$, and $\overline{\Lambda_0(z)} = \Lambda_{-1}(\bar{z})$, we conclude that the solutions corresponding to the special cases (i) and (ii) are just the three zeros t_0, z_0, and z_{-1}, with $\bar{z}_0 = z_{-1}$, of $\Omega(z)$, and thus if we write

$$(t_0 - z)(z_0 - z)(z_{-1} - z) = \Omega(z)\, H^{-1}(z), \qquad a \in (-\infty, 0), \quad b > 0, \tag{26}$$

then $H(z)$ must be the (suitably normalized) canonical solution of the Riemann problem

$$H^+(t) = G(t)\, H^-(t), \qquad t \in (-\infty, -b)\ U(0, \infty), \tag{27}$$

where

$$G(t) = \frac{\Omega^+(t)}{\Omega^-(t)}. \tag{28}$$

The limiting values of $\Omega(z)$ can, of course, be computed readily from Eq. (24):

$$\begin{gathered}\Omega^{\pm}(t) = \ln|a| + \ln|t+b| - t - \ln|t| \pm \pi i, \\ t \in (-\infty, -b)\, U(0, \infty), \quad b > 0.\end{gathered} \tag{29}$$

The Riemann problem defined by Eq. (27) can be solved to yield

$$\begin{aligned}H(z) = {} & \frac{1}{z(z+b)} \\ & \times \exp\left[-\frac{1}{\pi}\int_b^\infty \arg\Omega^+(-t)\frac{dt}{t+z} + \frac{1}{\pi}\int_0^\infty [\arg\Omega^+(t) - \pi]\frac{dt}{t-z}\right] \\ & \qquad\qquad b > 0,\end{aligned} \tag{30}$$

or alternatively

$$\begin{aligned}H(z) = {} & \frac{1}{z(z+b)}\exp\left[-\frac{1}{\pi}\int_{b/(b+1)}^1 \arg\Omega^+\left(\frac{\tau}{\tau-1}\right)\frac{d\tau}{\tau(1-\tau)+z(1-\tau)^2}\right. \\ & \left. + \frac{1}{\pi}\int_0^1\left[\arg\Omega^+\left(\frac{\tau}{1-\tau}\right) - \pi\right]\frac{d\tau}{\tau(1-\tau)-z(1-\tau)^2}\right], \qquad b > 0;\end{aligned} \tag{31}$$

here $\arg\Omega^+(t)$ is continous on each of the intervals $(-\infty, -b]$ and $[0, \infty)$, with $\arg\Omega^+(-\infty) = 0$ and $\arg\Omega^+(0) = 0$. Since $H(z)$ is now established, Eq. (26) can be evaluated at three convenient values, say $z = \gamma$, $z = \eta$, and $z = \xi$, off the cuts and the three resulting equations solved simultaneously to yield closed-form results for t_0, z_0, and z_{-1}. For the sake of brevity, we shall not explicitly list these final results.

In the event that $b < 0$, we need to modify slightly the foregoing results; however, since the analysis is so similar, we shall only list the relevant solutions. In analogy with Eq. (15), we find

$$z_{01} = -B(\gamma, \eta) - (B^2(\gamma, \eta) - C(\gamma, \eta))^{1/2}, \qquad a \notin (-\infty, 0], \quad b < 0, \tag{32a}$$

and

$$z_{02} = -B(\gamma, \eta) + (B^2(\gamma, \eta) - C(\gamma, \eta))^{1/2}, \qquad a \notin (-\infty, 0], \quad b < 0, \tag{32b}$$

where

$$K(z) = -(z + b)\,\Lambda_0(z) \exp\left[\frac{1}{2\pi i}\int_b^0 \log G_0(-\tau)\frac{d\tau}{\tau + z}\right], \qquad a \notin (-\infty, 0], \quad b < 0, \tag{33}$$

is to be used in Eqs. (16) and (17) to yield $B(\gamma, \eta)$ and $C(\gamma, \eta)$. Note that here $\log G_0(t)$ is continuous for $t \in (0, -b)$ and such that $\log G_0(-b) = 2\pi i$. In regard to Eq. (23), we find the equivalent expression for $b < 0$ to be

$$z_k = z + [\log a + \log(z + b) - z - \log z + 2k\pi i] \times \exp\left[\frac{1}{2\pi i}\int_b^0 \log G_k(-\tau)\frac{d\tau}{\tau + z}\right], \qquad \{a, -b, k\} \in D, \quad k \neq 0, \tag{34}$$

where $\log G_k(t)$ is continuous for $t \in (0, -b)$ and such that

$$\log G_k(0) = \log G_k(-b) = 0.$$

In addition, we now have

$$\Lambda_k^{\pm}(t) = \log a + \ln|t + b| - t - \ln t + (2k \pm 1)\pi i, \qquad t \in (0, -b), \quad b < 0. \tag{35}$$

Finally we need to consider the special case, for $b < 0$, that $a \in (-\infty, 0)$ and $k = 0$ or $k = -1$. Here we find that

$$\Lambda_0^-(t) = \Lambda_{-1}^+(t) = \omega(t), \qquad t \in (0, -b), \quad a \in (-\infty, 0), \tag{36}$$

where

$$\omega(t) = \ln|a| + \ln|t + b| - t - \ln t, \qquad t \in (0, -b), \quad a \in (-\infty, 0). \tag{37}$$

We note here that $\omega(t)$ has precisely one real zero, say $t_0 \in (0, -b)$, which may be determined by considering the function

$$\Omega(z) = \ln|a| + \log(z + b) - z - \log z - \pi i, \quad a \in (-\infty, 0), \quad b < 0, \tag{38}$$

where $\log(z + b)$ is that branch of the log-function in the plane cut along the real axis from $-b$ to ∞, such that $0 < \arg(z + b) < 2\pi$. Clearly $\Omega(z) = \Lambda_{-1}(z)$ for $y > 0$ and $\Omega(z) = \Lambda_0(z)$ for $y < 0$. Now $\Omega(z)$ has only one zero t_0 in the cut plane, and the limiting values are given by

$$\Omega^{\pm}(t) = \ln|a| + \ln|t + b| - t - \ln|t| \mp \pi i, \qquad t \in (-\infty, 0)\, U(-b, \infty), \quad b < 0. \tag{39}$$

Here we find the (appropriately normalized) solution of

$$H^+(t) = \frac{\Omega^+(t)}{\Omega^-(t)} H^-(t), \qquad t \in (-\infty, 0)\ U(-b, \infty), \quad b < 0, \tag{40}$$

can be written as

$$H(z) = \exp\left[-\frac{1}{\pi}\int_0^1 \arg \Omega^+\left(\frac{\tau}{\tau-1}\right) \frac{d\tau}{\tau(1-\tau) + z(1-\tau)^2} + \frac{1}{\pi}\int_{b/(b-1)}^1 \left[\arg \Omega^+\left(\frac{\tau}{1-\tau}\right) + \pi\right] \frac{d\tau}{\tau(1-\tau) - z(1-\tau)^2}\right], \qquad b < 0. \tag{41}$$

It therefore follows that

$$\frac{\Omega(z)}{t_0 - z} = H(z),$$

or

$$t_0 = z + \Omega(z)\, H^{-1}(z), \qquad a \in (-\infty, 0), \quad b < 0. \tag{42}$$

Finally for $a \in (-\infty, 0)$ we note that $\Lambda_0(z)$ and $\Lambda_{-1}(z)$ have zeros z_0 and z_{-1}, with $\bar{z}_0 = z_{-1}$, where $\operatorname{Im} z_0 > 0$. The determination of this root is effected by considering the function

$$\Delta(z) = \ln|a| + \log(z+b) - z - \log z + \pi i, \qquad a \in (-\infty, 0), \quad b<0, \tag{43a}$$

$$= \Omega(z) + 2\pi i. \tag{43b}$$

The function $\Delta(z) = \Lambda_0(z)$ for $y > 0$ and has only one zero which clearly is z_0. In a manner similar to the above we deduce that

$$z_0 = z + \Delta(z)\, \hat{H}^{-1}(z), \qquad a \in (-\infty, 0), \quad b < 0, \tag{44}$$

where

$$\hat{H}(z) = \exp\left\{-\frac{1}{2\pi i}\int_0^1 \log \hat{G}\left(\frac{\tau}{\tau-1}\right) \frac{d\tau}{\tau(1-\tau) + z(1-\tau)^2} + \frac{1}{2\pi i}\int_{b/(b-1)}^1 \log \hat{G}\left(\frac{\tau}{1-\tau}\right) \frac{d\tau}{\tau(1-\tau) - z(1-\tau)^2}\right\}, \tag{45}$$

with

$$\hat{G}(t) = \frac{\Delta^+(t)}{\Delta^-(t)}. \tag{46}$$

Although the required analysis may prove tedious, it is clear that the solutions given here can be generalized to include complex b and that, in general, transcendental equations of the form

$$P_1(z)\, e^z = P_2(z)$$

where $P_1(z)$ and $P_2(z)$ are polynomials, can be solved by the method discussed here.

Acknowledgments

The authors are grateful to A. R. Burkart for numerically evaluating our solutions, for all cases, with an accuracy of six significant figures. One of the authors (CES) is grateful to Professor I. Sneddon, Glasgow University, and the Science Research Council (grant B/SR/9483) for their kind hospitality and support of this work. This investigation was also partially supported by Joint Advisory Group Grant AFOSR-69-1779.

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