J. Quant. Spectrosc. Radiat. Transfer. Vol. 14, pp. 259–271. Pergamon Press 1974. Printed in Great Britain.

NON-GRAY RADIATIVE TRANSFER

Y. YENER, M. N. ÖZIŞIK and C. E. SIEWERT
Departments of Mechanical and Aerospace and Nuclear Engineering, North Carolina State University, Raleigh, North Carolina 27607, U.S.A.

(Received 23 July 1973)

Abstract—The normal-mode-expansion technique is used to establish the solution of the Milne problem basic to a generalized equation of radiative transfer. The non-gray model used includes the effects of absorption, scattering and losses due to photo-electric ionizations and collisions of the second kind. Accurate numerical results are presented for such physical quantities as the extrapolation distance, the integrated Planck function and the angular distribution of the exit intensity for selected values of the basic parameters.

1. INTRODUCTION

THE PICKET-fence model originally discussed by CHANDRASEKHAR[1] in an analysis of non-gray radiative transfer for astrophysical applications is based on the concept of representing the absorption and scattering coefficients by a set of discrete values over the entire frequency spectrum. CHANDRASEKHAR[1] and MÜNCH[2] reported approximate solutions for the Milne problem in the picket-fence model. More recently, SIEWERT and ZWEIFEL,[3] utilizing the CASE[4] technique, developed the formalism for the exact solution of the equation of radiative transfer in the picket-fence model by assuming LTE and radiative equilibrium and by neglecting the effects of scattering. This exact method of solution was applied by SIMMONS and FERZIGER,[5] SIEWERT and ÖZIŞIK,[6] BOND and SIEWERT,[7] and REITH *et al.*,[8] to study radiative transfer in the picket-fence model. However, in all of these studies,[3, 5–8] the transfer matrix in the equation of radiative transfer had a simple form, that is, the determinant of the transfer matrix vanished, because the scattering term in the defining equation was neglected. The elementary solutions of the more general equation of transfer have been summarized by SIEWERT and SHIEH[9] for isotropic scattering, and the related half-range orthogonality theorem and normalization integrals have been established by SIEWERT and ISHIGURO.[10]

In the present analysis, we consider the generalized equation of radiative transfer in the two-group picket-fence model with absorption, isotropic scattering and losses due to photo-electric ionizations and collisions of the second kind. The elementary solutions of the resulting equations are then used to develop the solution of the Milne problem.

2. ANALYSIS

We consider here the following form of the equation of radiative transfer for isotropic scattering:[11]

$$\mu \frac{\partial}{\partial x} I_\nu(x, \mu) + (\kappa_\nu + \sigma_\nu) I_\nu(x, \mu) = (\kappa_\nu + \varepsilon_\nu \sigma_\nu) B_\nu[T(x)] + \frac{1}{2} \sigma_\nu (1 - \varepsilon_\nu) \int_{-1}^{1} I_\nu(x, \mu') \, d\mu', \quad (1)$$

where $I_\nu(x, \mu)$ is the spectral radiation intensity, κ_ν and σ_ν are respectively the spectral absorption and scattering coefficients, $B_\nu[T(x)]$ is the Planck function at the local temperature $T(x)$, and μ is the direction cosine of the propagating radiation (as measured from the positive x axis). Here, the coefficient $\varepsilon_\nu \ll 1$ allows for the possibility that a certain amount of thermal emission may be associated with the scattering coefficient; the origin of this process of selective scattering and reemission may be due to photo-electric ionizations and collisions of the second kind. For $\varepsilon_\nu = 0$ equation (1) simplifies to the standard form of the equation of radiative transfer.

We now assume that the entire frequency spectrum is divided into two regions $\Delta\nu_i$, $i = 1, 2$, in each of which κ_ν, σ_ν and ε_ν take constant values κ_i, σ_i and ε_i. Integration of equation (1) over the region $\Delta\nu_i$ yields

$$\mu \frac{\partial}{\partial x} I_i(x, \mu) + (\kappa_i + \sigma_i) I_i(x, \mu) = \frac{1}{2} \sum_{j=1}^{2} \left[\frac{(\kappa_i + \varepsilon_i \sigma_i)\omega_i}{\sum_{s=1}^{2} (\kappa_s + \varepsilon_s \sigma_s)\omega_s} (\kappa_j + \varepsilon_j \sigma_j) + \delta_{ij} \sigma_j (1 - \varepsilon_j) \right] \int_{-1}^{1} I_j(x, \mu') \, d\mu'. \quad (2)$$

Here we have utilized the condition of radiative equilibrium

$$\int_0^\infty (\kappa_\nu + \varepsilon_\nu \sigma_\nu) B_\nu[T(x)] \, d\nu = \frac{1}{2} \int_0^\infty (\kappa_\nu + \varepsilon_\nu \sigma_\nu) \int_{-1}^{1} I_\nu(x, \mu) \, d\mu \, d\nu \quad (3)$$

and defined

$$I_i(x, \mu) = \int_{\Delta\nu_i} I_\nu(x, \mu) \, d\nu \quad (4)$$

and

$$\omega_i = \frac{\pi}{\bar{\sigma} T^4(x)} \int_{\Delta\nu_i} B_\nu[T(x)] \, d\nu = \frac{1}{B(x)} \int_{\Delta\nu_i} B_\nu[T(x)] \, d\nu, \quad (5)$$

where $B(x)$ is the integrated Planck function and $\bar{\sigma}$ is the Stefan–Boltzmann constant. We note that ω_1 and $\omega_2 = 1 - \omega_1$ are, in general, functions of the space variable x; however, following the works previously mentioned, we consider that ω_1 and ω_2 are constants.

Introducing an optical variable

$$d\tau = (\kappa_2 + \sigma_2) \, dx, \quad (6)$$

we can write equation (2) in the form

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{\Sigma} \mathbf{I}(\tau, \mu) = \mathbf{Q} \int_{-1}^{1} \mathbf{I}(\tau, \mu') \, d\mu'. \quad (7)$$

Here $\mathbf{I}(\tau, \mu)$ is a two-component vector with elements $I_1(\tau, \mu)$ and $I_2(\tau, \mu)$, while

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix} \quad (8)$$

and the elements of the 2×2 transfer matrix $\mathbf{Q}$ are

$$q_{ij} = \frac{1}{2} \frac{\sigma_{ij}}{\kappa_2 + \sigma_2}, \quad (9)$$

where

$$\sigma_{ij} = \frac{(\kappa_i + \varepsilon_i \sigma_i)(\kappa_j + \varepsilon_j \sigma_j)\omega_i}{\sum_{s=1}^{2} (\kappa_s + \varepsilon_s \sigma_s)\omega_s} + \delta_{ij}\sigma_j(1 - \varepsilon_j) \tag{10}$$

and

$$\sigma = \frac{\kappa_1 + \sigma_1}{\kappa_2 + \sigma_2}, \qquad \sigma > 1. \tag{11}$$

Without loss of generality σ is taken to be greater than unity. We note that the transfer matrix **Q** in equation (7) is not symmetric; however, the equation can be transformed to a form with a symmetric transfer matrix. If we let

$$\mathbf{P} = \begin{bmatrix} \left(\dfrac{\omega_2}{\omega_1}\right)^{1/2} & 0 \\ 0 & 1 \end{bmatrix}, \tag{12}$$

then we can pre-multiply equation (7) by **P** to yield

$$\mu \frac{\partial}{\partial \tau} \mathbf{\Psi}(\tau, \mu) + \mathbf{\Sigma}\mathbf{\Psi}(\tau, \mu) = \mathbf{C} \int_{-1}^{1} \mathbf{\Psi}(\tau, \mu')\, d\mu', \tag{13}$$

where

$$\mathbf{\Psi}(\tau, \mu) = \mathbf{PI}(\tau, \mu), \tag{14}$$

$$\mathbf{C} = \mathbf{PQP}^{-1}, \tag{15}$$

and the elements of the new transfer matrix **C** are given by

$$c_{ij} = \frac{\lambda_i \lambda_j (\omega_i \omega_j)^{1/2}}{2\sum_{s=1}^{2} \lambda_s \omega_s} + \delta_{ij} \frac{(\sigma\delta_{1i} + \delta_{2j}) - \lambda_i}{2}, \tag{16}$$

where λ_i is defined as

$$\lambda_i = \frac{\kappa_i + \varepsilon_i \sigma_i}{\kappa_2 + \sigma_2}, \qquad i = 1 \text{ and } 2. \tag{17}$$

We note that $0 < \lambda_1 \le \sigma$ and $0 < \lambda_2 \le 1$ and that **C** is symmetric. In addition,

$$C = \det \mathbf{C} = \frac{\sigma(1 - \lambda_2)\lambda_1 \omega_1 + (\sigma - \lambda_1)\lambda_2 \omega_2}{4(\lambda_1 \omega_1 + \lambda_2 \omega_2)}. \tag{18}$$

Invoking the definitions of the λ_i's and σ, we note that

$$\det \mathbf{C} \ge 0 \tag{19}$$

and

$$\det[\mathbf{\Sigma} - 2\mathbf{C}] = \sigma + 4C - 2c_{11} - 2\sigma c_{22} = 0. \tag{20}$$

We note that SIEWERT and ZWEIFEL[3] have expressed a general solution of equation (13) as

$$\Psi(\tau,\mu) = A_+\Phi_+ + A_-\Phi_-(\tau,\mu) + \int_{①} [A_1^{(1)}(\eta)\Phi_1^{(1)}(\eta,\mu) + A_2^{(1)}(\eta)\Phi_2^{(1)}(\eta,\mu)]\exp(-\tau/\eta)\,d\eta + \int_{②} A^{(2)}(\eta)\Phi^{(2)}(\eta,\mu)\exp(-\tau/\eta)\,d\eta, \quad (21)$$

where regions ① and ② respectively refer to $\eta \in (-1/\sigma, 1/\sigma)$ and $\eta \in (-1, -1/\sigma)U(1/\sigma, 1)$, and A_+, A_-, $A_1^{(1)}(\eta)$, $A_2^{(1)}(\eta)$ and $A^{(2)}(\eta)$ are the expansion coefficients to be determined once appropriate boundary conditions have been specified. The continuum eigenvectors required in equation (21) can be written as

$$\Phi_1^{(1)}(\eta,\mu) = \begin{bmatrix} c_{11}\eta \dfrac{P}{\sigma\eta-\mu} + \lambda_{11}(\eta)\delta(\sigma\eta-\mu) \\ c_{21}\eta \dfrac{P}{\eta-\mu} + \lambda_{21}(\eta)\delta(\eta-\mu) \end{bmatrix}, \quad (22a)$$

$$\Phi_2^{(1)}(\eta,\mu) = \begin{bmatrix} c_{12}\eta \dfrac{P}{\sigma\eta-\mu} + \lambda_{12}(\eta)\delta(\sigma\eta-\mu) \\ c_{22}\eta \dfrac{P}{\eta-\mu} + \lambda_{22}(\eta)\delta(\eta-\mu) \end{bmatrix}, \quad (22b)$$

and

$$\Phi^{(2)}(\eta,\mu) = \begin{bmatrix} \dfrac{c_{12}\eta}{\sigma\eta-\mu} \\ \eta f(\eta) \dfrac{P}{\eta-\mu} + \lambda(\eta)\delta(\eta-\mu) \end{bmatrix}, \quad (22c)$$

where

$$f(\eta) = c_{22} - 2C\eta \tanh^{-1}\left(\frac{1}{\sigma\eta}\right), \quad (23a)$$

$$\lambda(\eta) = \det \boldsymbol{\lambda}(\eta), \quad (23b)$$

and

$$\boldsymbol{\lambda}(\eta) = \mathbf{I} + \eta P \int_{-1}^{1} \boldsymbol{\psi}(\mu)\frac{d\mu}{\mu-\eta}. \quad (24)$$

Here the characteristic matrix $\boldsymbol{\psi}(\mu)$ is

$$\boldsymbol{\psi}(\mu) = \begin{bmatrix} \theta(\mu) & 0 \\ 0 & 1 \end{bmatrix}\mathbf{C} = \boldsymbol{\theta}(\mu)\mathbf{C} \quad (25)$$

with $\theta(\mu) = 1$ for $\mu \in (-1/\sigma, 1/\sigma)$ and $\theta(\mu) = 0$, otherwise. In addition, the symbol P is used to denote that ensuing integrals over μ or η are to be evaluated in the Cauchy principal-value sense, $\delta(x)$ is the Dirac delta functional and $\mathbf{I}$ is the unit matrix.

Regarding the discrete spectrum $\eta \notin (-1, 1)$, we note that the number of zeros of the dispersion function $\Lambda(z) = \det \mathbf{\Lambda}(z)$, with

$$\mathbf{\Lambda}(z) = \mathbf{I} + z \int_{-1}^{1} \mathbf{\psi}(\mu) \frac{d\mu}{\mu - z}, \tag{26}$$

has been discussed by SIEWERT and SHIEH.[9] Invoking the results given by equations (19) and (20), we conclude that the dispersion function for the considered problem has either two zeros at infinity or two zeros at infinity plus two real zeros. In the present analysis we focus our attention on the situation where there are only two roots at infinity, *i.e.* $c_{22} > 2C\tanh^{-1}(1/\sigma)$. Thus we write the discrete eigenvectors as

$$\mathbf{\Phi}_+ = \begin{bmatrix} \dfrac{c_{12}}{\sigma} \\ c_{22} - \dfrac{2C}{\sigma} \end{bmatrix} \quad \text{and} \quad \mathbf{\Phi}_-(\tau, \mu) = \begin{bmatrix} \dfrac{c_{12}}{\sigma^2}(\tau\sigma - \mu) \\ \left(c_{22} - \dfrac{2C}{\sigma}\right)(\tau - \mu) \end{bmatrix}. \tag{27}$$

Basic to the determination of the expansion coefficients for half-space applications is the half-range completeness theorem[12] which states that an arbitrary Hölder function $\mathbf{F}(\mu)$ may be expressed as

$$\mathbf{F}(\mu) = A_+ \mathbf{\Phi}_+ + \int_0^{1/\sigma} [A_1^{(1)}(\eta)\mathbf{\Phi}_1^{(1)}(\eta, \mu) + A_2^{(1)}(\eta)\mathbf{\Phi}_2^{(1)}(\eta, \mu)]\, d\eta$$
$$+ \int_{1/\sigma}^{1} A^{(2)}(\eta)\mathbf{\Phi}^{(2)}(\eta, \mu)\, d\eta, \qquad \mu \in (0, 1), \tag{28}$$

where the expansion coefficients can be determined by utilizing the orthogonality theorem of SIEWERT and ISHIGURO:[10]

$$A_+ = \frac{1}{N_+} \int_0^1 \tilde{\mathbf{\Theta}}(\mu)\mathbf{F}(\mu)\mu\, d\mu, \tag{29a}$$

$$A_\alpha^{(1)}(\eta) = \frac{1}{N^{(1)}(\eta)} \int_0^1 \tilde{\mathbf{\Theta}}_\alpha^{(1)}(\eta, \mu)\mathbf{F}(\mu)\mu\, d\mu, \qquad \alpha = 1 \text{ and } 2, \tag{29b}$$

and

$$A^{(2)}(\eta) = \frac{1}{N^{(2)}(\eta)} \int_0^1 \tilde{\mathbf{\Theta}}^{(2)}(\eta, \mu)\mathbf{F}(\mu)\mu\, d\mu, \tag{29c}$$

where the normalization factors are given by

$$N_+ = \frac{4}{3}\left[\frac{c_{12}^2}{\sigma^3} + \left(c_{22} - \frac{2C}{\sigma}\right)^2\right], \tag{30a}$$

$$N^{(1)}(\eta) = \eta\Lambda^+(\eta)\Lambda^-(\eta), \qquad \eta \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right), \tag{30b}$$

and

$$N^{(2)}(\eta) = \eta\Lambda^+(\eta)\Lambda^-(\eta), \qquad \eta \in \left(-1, -\frac{1}{\sigma}\right) U \left(\frac{1}{\sigma}, 1\right). \tag{30c}$$

Here $\Lambda^{\pm}(\eta)$ denote the limiting values as the branch cut of $\Lambda(z)$ is approached from above (+) and below (−); explicitly we can write

$$\Lambda^{\pm}(\eta) = \lambda_{11}(\eta)\lambda_{22}(\eta) - \lambda_{12}(\eta)\lambda_{21}(\eta) - (\pi\eta)^2 C\theta(\eta) \pm i\pi\eta\{\lambda_{11}(\eta)c_{22} + \lambda_{12}(\eta)c_{12} + \theta(\eta)[c_{11}\lambda_{22}(\eta) - c_{12}\lambda_{21}(\eta)]\}, \quad (31)$$

where the elements of $\boldsymbol{\lambda}(\eta)$ are denoted by $\lambda_{ij}(\eta)$ and given by equation (24). In the foregoing equations, the superscript tilde is used to denote the transpose operation. In addition the functions $\boldsymbol{\Theta}(\mu)$, $\boldsymbol{\Theta}_\alpha^{(1)}(\eta, \mu)$ and $\boldsymbol{\Theta}^{(2)}(\eta, \mu)$ are given by

$$\boldsymbol{\Theta}(\mu) = 2\boldsymbol{\Sigma}^{-1}\mathbf{h}(\mu)\mathbf{C}\tilde{\mathbf{H}}_1\mathbf{U}, \quad (32a)$$

$$\boldsymbol{\Theta}_\alpha^{(1)}(\eta, \mu) = [\eta\mathbf{K}(\eta, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{C} + \boldsymbol{\delta}(\eta, \mu)\boldsymbol{\lambda}(\eta)]\mathbf{V}_\alpha(\eta), \qquad \alpha = 1 \text{ and } 2, \quad (32b)$$

and

$$\boldsymbol{\Theta}^{(2)}(\eta, \mu) = [\eta\mathbf{K}(\eta, \mu)\mathbf{h}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{C} + \boldsymbol{\delta}(\eta, \mu)\boldsymbol{\lambda}(\eta)]\mathbf{U}^{(2)}(\eta), \quad (32c)$$

where

$$\mathbf{K}(\eta, \mu) = \begin{bmatrix} \dfrac{P}{\sigma\eta - \mu} & 0 \\ 0 & \dfrac{P}{\eta - \mu} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\delta}(\eta, \mu) = \begin{bmatrix} \delta(\sigma\eta - \mu) & 0 \\ 0 & \delta(\eta - \mu) \end{bmatrix}. \quad (33)$$

Further,

$$\mathbf{V}_1(\eta) = N_{22}(\eta)\mathbf{U}_1^{(1)}(\eta) - N_{12}(\eta)\mathbf{U}_2^{(1)}(\eta), \quad (34a)$$

$$\mathbf{V}_2(\eta) = N_{11}(\eta)\mathbf{U}_2^{(1)}(\eta) - N_{21}(\eta)\mathbf{U}_1^{(1)}(\eta), \quad (34b)$$

$$\mathbf{U}^{(2)}(\eta) = \begin{bmatrix} -\Lambda_{12}(\eta) \\ \Lambda_{11}(\eta) \end{bmatrix}, \quad (35a)$$

$$\mathbf{U}_1^{(1)}(\eta) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{U}_2^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (35b)$$

To complete equations (34), we also require

$$N_{11}(\eta) = 1 - 4c_{11}\eta\Gamma(\sigma\eta) + 4\eta^2[c_{11}{}^2\Gamma^2(\sigma\eta) + c_{12}c_{21}\Gamma^2(\eta)] + \pi^2\eta^2(c_{11}{}^2 + c_{12}c_{21}), \quad (36a)$$

$$N_{ij}(\eta) = c_{ij}[4c_{11}\eta^2\Gamma^2(\sigma\eta) + 4c_{22}\eta^2\Gamma^2(\eta) - 2\eta\Gamma(\sigma\eta) + 2\eta\Gamma(\eta) + \pi^2\eta^2(c_{11} + c_{22})], \qquad i \neq j, \quad (36b)$$

and

$$N_{22}(\eta) = 1 - 4c_{22}\eta\Gamma(\eta) + 4\eta^2[c_{22}{}^2\Gamma^2(\eta) + c_{12}c_{21}\Gamma^2(\sigma\eta)] + \pi^2\eta^2(c_{22}{}^2 + c_{12}c_{21}), \quad (36c)$$

where we have used the abbreviation $\Gamma(x)$ for $\tanh^{-1}(x)$. In the foregoing equations the $\mathbf{H}$ matrix is the unique[12] solution of

$$\mathbf{H}(\mu) = \mathbf{I} + \mu\mathbf{H}(\mu)\mathbf{C}\int_0^1 \tilde{\mathbf{H}}(\mu')\boldsymbol{\theta}(\mu')\frac{d\mu'}{\mu + \mu'} \quad (37a)$$

and

$$\mathbf{U} = \tilde{\mathbf{H}}_0 \mathbf{U}. \tag{37b}$$

Here the moments $\mathbf{H}_n$, $n = 0, 1, 2 \ldots$, are defined as

$$\mathbf{H}_n = \int_0^1 \boldsymbol{\psi}(\mu)\mathbf{H}(\mu)\mu^n \, d\mu, \qquad n = 0, 1, 2, \ldots, \tag{37c}$$

and $\mathbf{U}$ is the normalization integral of the discrete eigenvector $\mathbf{\Phi}_+$:

$$\mathbf{U} = \int_{-1}^{1} \mathbf{\Phi}_+ \, d\mu = 2\mathbf{\Phi}_+. \tag{38}$$

Finally $\mathbf{h}(\mu)$ is defined as

$$\mathbf{h}(\mu) = \begin{bmatrix} H_{11}\left(\dfrac{\mu}{\sigma}\right) & H_{12}\left(\dfrac{\mu}{\sigma}\right) \\ H_{21}(\mu) & H_{22}(\mu) \end{bmatrix}, \tag{39}$$

where the elements of $\mathbf{H}(\mu)$ are denoted by $H_{ij}(\mu)$.

3. THE MILNE PROBLEM

Having completed the required analytical formalism, we now proceed to solve a typical half-space problem. We consider the Milne problem and thus seek a diverging (as $\tau \to \infty$) solution to the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{\Sigma I}(\tau, \mu) = \mathbf{Q} \int_{-1}^{1} \mathbf{I}(\tau, \mu') \, d\mu', \qquad 0 \le \tau < \infty, \tag{40a}$$

subject to the boundary conditions

$$\mathbf{I}(0, \mu) = \mathbf{0}, \qquad \mu \in (0, 1), \tag{40b}$$

and

$$\lim_{\tau \to \infty} \mathbf{I}(\tau, \mu) e^{-\tau} = \mathbf{0}. \tag{40c}$$

Here $\mathbf{I}(\tau, \mu)$ is the radiation intensity vector; $\mathbf{\Sigma}$ and $\mathbf{Q}$ are defined in terms of the basic parameters by equations (8) and (9). To transform equation (40a) to a form with a symmetric transfer matrix we introduce

$$\mathbf{\Psi}(\tau, \mu) = \mathbf{PI}(\tau, \mu) \tag{41}$$

where $\mathbf{P}$ is given by equation (12). Thus, with $\mathbf{C} = \tilde{\mathbf{C}}$, we consider

$$\mu \frac{\partial}{\partial \tau} \mathbf{\Psi}(\tau, \mu) + \mathbf{\Sigma \Psi}(\tau, \mu) = \mathbf{C} \int_{-1}^{1} \mathbf{\Psi}(\tau, \mu') \, d\mu', \qquad 0 \le \tau < \infty, \tag{42a}$$

subject to the boundary conditions

$$\mathbf{\Psi}(0, \mu) = \mathbf{0}, \qquad \mu \in (0, 1), \tag{42b}$$

and

$$\lim_{\tau \to \infty} \mathbf{\Psi}(\tau, \mu) e^{-\tau} = \mathbf{0}. \tag{42c}$$

A properly diverging solution of equation (42a) can be written as

$$\begin{aligned}\mathbf{\Psi}(\tau, \mu) = {} & A_+\mathbf{\Phi}_+ + A_-\mathbf{\Phi}_-(\tau, \mu) \\ & + \int_0^{1/\sigma} [A_1^{(1)}(\eta)\mathbf{\Phi}_1^{(1)}(\eta, \mu) + A_2^{(1)}(\eta)\mathbf{\Phi}_2^{(1)}(\eta, \mu)]\exp(-\tau/\eta)\, d\eta \\ & + \int_{1/\sigma}^{1} A^{(2)}(\eta)\mathbf{\Phi}^{(2)}(\eta, \mu)\exp(-\tau/\eta)\, d\eta; \end{aligned} \tag{43}$$

constraining this solution to meet the boundary condition given by equation (42b), we find

$$\begin{aligned}\mu\mathbf{\Sigma}^{-1}\mathbf{\Phi}_+ = \tau_0\mathbf{\Phi}_+ + \int_0^{1/\sigma} & [B_1^{(1)}(\eta)\mathbf{\Phi}_1^{(1)}(\eta, \mu) + B_2^{(1)}(\eta)\mathbf{\Phi}_2^{(1)}(\eta, \mu)]\, d\eta \\ & + \int_{1/\sigma}^{1} B^{(2)}(\eta)\mathbf{\Phi}^{(2)}(\eta, \mu)\, d\eta, \qquad \mu \in (0, 1), \end{aligned} \tag{44}$$

where we have defined

$$\tau_0 = \frac{A_+}{A_-}, \quad B_\alpha^{(1)}(\eta) = \frac{A_\alpha^{(1)}(\eta)}{A_-}, \quad \alpha = 1 \text{ and } 2, \quad \text{and } B^{(2)}(\eta) = \frac{A^{(2)}(\eta)}{A_-}. \tag{45}$$

Here we can fix the arbitrary constant A_- by imposing a normalization condition:

$$2\begin{bmatrix}1\\1\end{bmatrix}^T \int_{-1}^{1} \mathbf{I}(\tau, \mu)\mu\, d\mu = -F, \tag{46}$$

where F is the flux constant and the superscript T is used to denote the transpose operation. We can now use equations (41) and (43) to find

$$A_- = \frac{3\sigma^2 F}{4\left[\left(\dfrac{\omega_1}{\omega_2}\right)^{1/2} c_{12} + \sigma^2 c_{22} - 2\sigma C\right]}. \tag{47}$$

To complete our solution, we can now solve equation (44), by making use of equations (29), to obtain

$$\tau_0 = \frac{2}{N_+}\tilde{\mathbf{U}}\mathbf{H}_1\mathbf{C}\tilde{\mathbf{H}}_2\mathbf{U}, \tag{48}$$

$$B_\alpha^{(1)}(\eta) = -\frac{\eta}{N^{(1)}(\eta)}\tilde{\mathbf{V}}_\alpha(\eta)\mathbf{C}\mathbf{H}^{-1}(\eta)\mathbf{H}_1\mathbf{U}, \qquad \alpha = 1 \text{ and } 2, \tag{49}$$

and

$$B^{(2)}(\eta) = -\frac{\eta}{N^{(2)}(\eta)}\tilde{\mathbf{U}}^{(2)}(\eta)\mathbf{C}\mathbf{H}^{-1}(\eta)\mathbf{H}_1\mathbf{U}. \tag{50}$$

Since the complete radiation intensity $\mathbf{I}(\tau, \mu)$ is now expressed in terms of the $\mathbf{H}$ matrix and known quantities, we can readily determine other physical quantities of interest. For example, $\mathbf{I}(\tau, \mu)$ can be entered into equation (3) to yield the black-body radiation function:

$$B(\tau) = \frac{A_-}{2(\lambda_1\omega_1 + \lambda_2\omega_2)}\left\{2(\tau_0 + \tau)\left[\lambda_1 \frac{c_{12}}{\sigma}\left(\frac{\omega_1}{\omega_2}\right)^{1/2} + \lambda_2\left(c_{22} - \frac{2C}{\sigma}\right)\right]\right.$$
$$\left. + \begin{bmatrix} \lambda_1\left(\dfrac{\omega_1}{\omega_2}\right)^{1/2} \\ \lambda_2 \end{bmatrix}^T \int_0^1 \mathbf{B}(\eta)\exp(-\tau/\eta)\,\mathrm{d}\eta\right\}, \quad (51)$$

where

$$\mathbf{B}(\eta) = \theta(\eta)\begin{bmatrix} B_1^{(1)}(\eta) \\ B_2^{(1)}(\eta) \end{bmatrix} + [1 - \theta(\eta)]B^{(2)}(\eta)\mathbf{U}^{(2)}(\eta). \quad (52)$$

The angular distribution of the radiation emerging from the half space follows at once from equations (41) and (43) after we set $\tau = 0$ and consider $\mu < 0$. Alternatively we can use equation (80b) of SIEWERT and ISHIGURO[10] to deduce that

$$\mathbf{I}(0, -\mu) = A_-\mathbf{P}^{-1}\mathbf{\Sigma}^{-1}\mathbf{h}(\mu)\mathbf{C}\tilde{\mathbf{H}}_1\mathbf{U}, \qquad \mu \in (0, 1). \quad (53)$$

It is apparent from the foregoing analysis that an accurate computation of the $\mathbf{H}$ matrix is basic to a calculation of the various physical quantities of interest here. We have found that a straightforward iterative calculation of $\mathbf{H}(\mu)$ from equation (37a) subject to equation (37b), does not converge very quickly. However we have found that the method discussed by KRIESE and SIEWERT[13] can be used here to accelerate our iterative calculation. If we write

$$\mathbf{H}(z) = \mathbf{T}\mathbf{B}(z)\mathbf{L}(z)\mathbf{T}^{-1}, \quad (54)$$

where

$$\mathbf{B}(z) = \begin{bmatrix} 1 + z & 0 \\ 0 & 1 \end{bmatrix}, \quad (55)$$

and

$$\mathbf{T} = a\begin{bmatrix} c_{12} & -2\left(c_{22} - \dfrac{2C}{\sigma}\right) \\ \left(c_{22} - \dfrac{2C}{\sigma}\right) & 2\dfrac{c_{12}}{\sigma} \end{bmatrix}, \quad (56a)$$

with

$$a = \frac{1}{\sqrt{2}}\left[\frac{c_{12}^{\,2}}{\sigma} + \left(c_{22} - \frac{2C}{\sigma}\right)^2\right]^{-1/2}, \quad (56b)$$

we can enter equation (54) into equation (37a) to obtain, after using equation (37b),

$$\mathbf{L}(\mu) = \mathbf{I} + \mu\mathbf{L}(\mu)\mathbf{\Delta}\int_0^1 \tilde{\mathbf{L}}(\mu')\mathbf{R}(\mu')\frac{\mathrm{d}\mu'}{\mu' + \mu}, \quad (57)$$

where

$$\mathbf{R}(\mu) = \mathbf{B}(\mu)\tilde{\mathbf{T}}\boldsymbol{\theta}(\mu)\mathbf{T}\mathbf{B}(-\mu), \tag{58}$$

and

$$\boldsymbol{\Delta} = \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix}. \tag{59}$$

We note that a solution of equation (57) can now be computed iteratively and that the iterative procedure converges very quickly, as compared to the original **H**-matrix equation. Of course, having found $\mathbf{L}(\mu)$ we can obtain an **H** matrix immediately from equation (54). To ensure that the **H** matrix so computed is, in fact, the correct one, equations (37a) and (37b) must now be verified. Finally we wish to remark that this particular procedure for computing the **H** matrix is restricted to the case considered here, *viz* $\Lambda(z)$ has only one pair of zeros, at infinity.

4. NUMERICAL RESULTS

Since the complete solution of the Milne problem has been established, we now wish to report some numerical results which follow immediately from the foregoing analysis, once the **H** matrix has been computed. All of our calculations were performed in double-precision arithmetic on an **IBM** 360/75 machine. To evaluate the **H** matrix from equations (54) and (57), a Gaussian quadrature scheme was used to represent the integral in equation (57). In regard to equation (57), the integration interval [0, 1] was subdivided into $[0, 1/\sigma]U[1/\sigma, 1]$ and a 40-point Gaussian integration scheme was used in each subinterval. In Table 1, we list the computed **H** matrix for the case $\sigma = 5{\cdot}0$, $\lambda_1 = 2{\cdot}5$, $\lambda_2 = 0{\cdot}5$, and $\omega_1 = 0{\cdot}4$.

Table 1. The **H** matrix for the case $\sigma = 5{\cdot}0, \lambda_1 = 2{\cdot}5, \lambda_2 = 0{\cdot}5$ and $\omega_1 = 0{\cdot}4$

μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0·0	1·0	0·0	0·0	1·0
0·1	1·62598	0·36601	0·09104	1·15565
0·2	2·03356	0·80269	0·16098	1·29237
0·3	2·37056	1·27512	0·22393	1·42671
0·4	2·66839	1·76768	0·28286	1·56027
0·5	2·94153	2·27266	0·33918	1·69347
0·6	3·19791	2·78581	0·39369	1·82648
0·7	3·44236	2·30463	0·44689	1·95936
0·8	3·67797	3·82750	0·49911	2·09215
0·9	3·90684	4·35339	0·55057	2·22488
1·0	4·13046	4·88157	0·60143	2·35755

Once $\mathbf{H}(\mu)$, $\mu \in [0, 1]$, was established numerically, we evaluated, in a straightforward manner, equations (48), (49), and (50) to complete the desired solution. In Table 2, we list our results for τ_0 for several of the considered cases, and in Table 3 we compare our results with the predictions of three approximate solutions.

Table 2. The extrapolation distance τ_0

σ	λ_1	λ_2	ω_1	τ_0
5	2·5	0·5	0·2	0·655319
5	2·5	0·5	0·4	0·592670
5	2·5	0·5	0·6	0·511113
5	2·5	0·5	0·8	0·387300
5	1	1	0·2	0·657507
10	1	1	0·2	0·668369

Table 3. A comparison of approximate and exact results for the extrapolation distance τ_0, for the case $\omega_1 = 0{\cdot}2$ and $\lambda_1 = \lambda_2 = 1$

	τ_0			
σ	Diffusion theory*	Approximate I†	Approximate II‡	Exact
5	0·61194	0·53228	0·61668	0·657507
10	0·62180	0·54106	0·65413	0·668369

* Chandrasekhar[1]
† Münch's,[2] first approximation
‡ Münch's,[2] second approximation

Table 4. The integrated black-body radiation intensity

	$B(\tau)/F$			
	$\sigma = 5{\cdot}0$, $\lambda_1 = 2{\cdot}5$ and $\lambda_2 = 0{\cdot}5$			
τ	$\omega_1 = 0{\cdot}2$	$\omega_1 = 0{\cdot}4$	$\omega_1 = 0{\cdot}6$	$\omega_1 = 0{\cdot}8$
0·00	0·34132	0·31225	0·31251	0·34195
0·02	0·38950	0·36955	0·37924	0·42336
0·04	0·42640	0·41397	0·43174	0·48827
0·10	0·51983	0·52842	0·56956	0·66173
0·20	0·65168	0·69363	0·77405	0·92611
0·30	0·76914	0·84305	0·96348	1·17741
0·40	0·87830	0·98278	1·14353	1·42108
0·50	0·98204	1·11573	1·31675	1·65922
0·60	1·08205	1·24368	1·48469	1·89300
0·80	1·27484	1·48912	1·80877	2·35031
1·00	1·46184	1·72540	2·12165	2·79725
1·20	1·64563	1·95609	2·42700	3·23658
1·40	1·82753	2·18328	2·72723	3·67027
1·60	2·00831	2·40826	3·02394	4·09975
2·00	2·36799	2·85443	3·61083	4·95007
3·00	3·26300	3·96138	5·06170	7·05067
4·00	4·15635	5·06516	6·50582	9·13850

In order to evaluate the black-body function $B(\tau)$, it is clear that the integral in equation (51) must be evaluated numerically. We found that the accuracy of our calculations could be improved by dividing the integration interval [0, 1] into subintervals in each of which we used a Gaussian scheme to distribute the nodal points. To help establish confidence in all of our calculations, the number of nodal points in our various integration schemes and the

Table 5. A comparison of approximate and exact results for the case $\omega_1 = 0{\cdot}2$ and $\lambda_1 = \lambda_2 = 1$

	$B(\tau)/F$			
	$\sigma = 5$		$\sigma = 10$	
τ	Approximate*	Exact	Approximate*	Exact
0·00	0·4097	0·42962	0·4308	0·42848
0·02	0·4348	0·46604	0·4603	0·46863
0·10	0·5295	0·57271	0·5664	0·58551
0·20	0·6400	0·68746	0·6856	0·70934
0·30	0·7454	0·79405	0·7972	0·82224
0·40	0·8475	0·89599	0·9039	0·92868
0·50	0·9472	0·99484	1·0072	1·03092
0·60	1·0450	1·09152	1·1079	1·13031
0·80	1·2364	1·28046	1·3041	1·32370
1·00	1·4238	1·46563	1·4956	1·51289
1·40	1·7910	1·82999	1·8709	1·88531
1·60	1·9726	2·01054	2·0564	2·07002
2·00	2·3331	2·37004	2·4251	2·43801
3·00	3·2287	3·26497	3·3418	3·35458
4·00	4·1220	4·15831	4·2568	4·26968

* Münch's,[2] second approximations.

number of subintervals used were increased until the reported results remained unchanged. In Table 4 we list sample results for $B(\tau)$ and in Table 5 we compare our results to an approximate solution.

Finally we have computed the exit distribution of radiation from equations (41) and (43), after setting $\tau = 0$ and considering $\mu < 0$, and also from equation (53). Since equation (53) does not require the expansion coefficients, the fact that the two equations yielded results identical to eight significant figures gives an additional degree of confidence in our results. In Table 6 we list our results for the exit distribution of radiation, and in Table 7 we compare our results to an approximate solution.

Table 6. The laws of darkening for the case $\sigma = 5$, $\lambda_1 = 2{\cdot}5$ and $\lambda_2 = 0{\cdot}5$

	$\dfrac{I_i(0, -\mu)}{I_1(0, -1) + I_2(0, -1)}$							
	$\omega_1 = 0{\cdot}2$		$\omega_1 = 0{\cdot}4$		$\omega_1 = 0{\cdot}6$		$\omega_1 = 0{\cdot}8$	
μ	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$
0·0	0·04516	0·25837	0·08662	0·20286	0·13524	0·15291	0·20531	0·09496
0·1	0·05325	0·34823	0·10442	0·28273	0·16538	0·21679	0·25378	0·13576
0·2	0·05927	0·41910	0·11808	0·34756	0·18910	0·27061	0·29275	0·17150
0·3	0·06460	0·48485	0·13042	0·40820	0·21090	0·32178	0·32910	0·20624
0·4	0·06952	0·54794	0·14198	0·46655	0·23156	0·37149	0·36399	0·24046
0·5	0·07415	0·60939	0·15296	0·52345	0·25143	0·42023	0·39788	0·27436
0·6	0·07856	0·66976	0·16351	0·57935	0·27068	0·46830	0·43104	0·30802
0·7	0·08279	0·72936	0·17371	0·63452	0·28944	0·51586	0·46361	0·34151
0·8	0·08687	0·78840	0·18362	0·68916	0·30778	0·56304	0·49570	0·37487
0·9	0·09084	0·84701	0·19329	0·74338	0·32577	0·60991	0·52738	0·40812
1·0	0·09471	0·90529	0·20273	0·79727	0·34345	0·65655	0·55871	0·44129

Table 7. A comparison of approximate and exact results for the laws of darkening with $\omega_1 = 0{\cdot}2$ and $\lambda_1 = \lambda_2 = 1$

$$\frac{I_1(0, -\mu) + I_2(0, -\mu)}{I_1(0, -1) + I_2(0, -1)}$$

	$\sigma = 5$		$\sigma = 10$	
μ	Diffusion theory*	Exact	Diffusion theory*	Exact
0·0	0·3499	0·31230	0·3397	0·29895
0·1	0·4209	0·40571	0·4173	0·39784
0·2	0·4883	0·48084	0·4873	0·47560
0·4	0·6188	0·61838	0·6193	0·61568
0·6	0·7468	0·74863	0·7475	0·74720
0·8	0·8737	0·87535	0·8742	0·87474
0·9	0·9369	0·93788	0·9372	0·93759
1·0	1·0000	1·00000	1·0000	1·00000

*Chandrasekhar.[1]

REFERENCES

1. S. Chandrasekhar, *Mon. Not. R. astr. Soc.* **96**, 21 (1935).
2. G. Münch, *Astrophys. J.* **104**, 87 (1946).
3. C. E. Siewert and P. F. Zweifel, *Annln Phys.* **36**, 61 (1966).
4. K. M. Case, *Annln Phys.* **9**, 1 (1960).
5. G. M. Simmons and J. H. Ferziger, *Int. J. Heat Mass Transfer* **11**, 1611 (1968).
6. C. E. Siewert and M. N Özişik, *Mon. Not. R. astr. Soc.* **146**, 351 (1969).
7. G. R. Bond and C. E. Siewert, *JQRST* **10**, 865 (1970).
8. R. J. Reith, Jr., C. E. Siewert and M. N. Özişik, *JQRST* **11**, 1441 (1971).
9. C. E. Siewert and P. S. Shieh, *J. nucl. Energy* **21**, 383 (1967).
10. C. E. Siewert and Y. Ishiguro, *J. nucl. Energy* **26**, 251 (1972)
11. S. Chandrasekhar, *Radiative Transfer*. Oxford University Press, London (1950); also Dover Publications, New York (1960).
12. C. E. Siewert, E. E. Burniston and J. T. Kriese, *J. nucl. Energy* **26**, 469 (1972).
13. J. T. Kriese and C. E. Siewert, *Astrophys. J.* **164**, 389 (1971).