Journal of Nuclear Energy, 1967, Vol. 21, pp. 383 to 392. Pergamon Press Ltd. Printed in Northern Ireland

TWO-GROUP TRANSPORT THEORY*

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(*Received* 9 *December* 1966)

Abstract—Case's method of singular normal modes is used to construct solutions to the two-group neutron transport equations in one dimension. Full-range completeness and orthogonality theorems for these eigensolutions are proved and the necessary normalization integrals are presented. In addition, functions orthogonal to the degenerate eigensolutions are developed so that all expansion coefficients can be found by simply taking scalar products. As an example of the method, the exact solution for the infinite-medium Green's function is obtained.

1. INTRODUCTION

IN ORDER to describe properly the behaviour of neutrons in a nuclear reactor system, one must be able to solve the general velocity-dependent neutron transport equation. The complexity of this equation leads one quite naturally to seek various ways to simplify the equation and to construct solutions thereof. DAVISON (1957) discusses in detail the formulation of the two-group transport equation and states the conditions under which the formulation should give a reasonably good approximation to the actual physical situation. In most reactor calculations one is unable to solve even this simplified version of the transport equation. As a consequence, the diffusion theory approximation is often invoked and the calculations are thus greatly simplified. The need, however, for exact solutions is very real since these solutions can be used as a standard against which the approximate solutions can be evaluated.

In a recent paper by SIEWERT and ZWEIFEL (1966), hereafter referred to as I, the eigensolutions of the two-group transport equation were presented and several completeness and orthogonality theorems proved. In that paper, however, the emphasis was on the application to the theory of radiative transfer in stellar atmospheres where the transfer matrix, **C**, in the transport equation had a simple form. The fact that the transfer matrix had a determinant which vanished, greatly reduced the complexity of the completeness and orthogonality theorems. In the present paper we remove the restrictions on **C** and prove the necessary full-space theorems for the set of eigensolutions of the two-group, one-dimensional transport equation and thus facilitate its application to neutron processes. In Section 2 these eigensolutions are presented and in Section 3 we prove the full-space completeness theorem. Section 4 is devoted to the proof of full-space orthogonality, and the calculation of the necessary normalization integrals is given. For the degenerate eigensolutions we also construct the necessary orthogonal functions. Finally in Section 5 we apply the method to determine exactly the infinite-medium Green's function.

2. EIGENVALUES AND EIGENSOLUTIONS

The two-group transport equation in one dimension for isotropic scattering can be written in the form (Siewert and Zweifel)

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Sigma\Psi}(x, \mu) = \mathbf{C} \int_{-1}^{1} \mathbf{\Psi}(x, \mu)\, \mathrm{d}\mu. \tag{1}$$

* This work was supported in part by the National Science Foundation.

Here $\mathbf{\Psi}(x, \mu)$ is a two-component vector, the elements of which are the respective group angular fluxes, i.e.

$$\mathbf{\Psi}(x, \mu) = \begin{bmatrix} \Psi_1(x, \mu) \\ \Psi_2(x, \mu) \end{bmatrix}. \tag{2}$$

The transfer matrix, $\mathbf{C}$, is arbitrary, with elements c_{ij}. By using the optical variable, x, we write the $\mathbf{\Sigma}$ matrix as

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma > 1. \tag{3}$$

We proceed as did CASE (1961) by seeking solutions to equation (1) of the form

$$\mathbf{\Psi}(x, \mu) = e^{-x/\eta}\mathbf{F}(\eta, \mu). \tag{4}$$

When this *ansatz* is substituted into equation (1), we obtain

$$\begin{bmatrix} \sigma\eta - \mu & 0 \\ 0 & \eta - \mu \end{bmatrix} \mathbf{F}(\eta, \mu) = \eta\mathbf{C} \int_{-1}^{1} \mathbf{F}(\eta, \mu)\, d\mu. \tag{5}$$

As discussed in I, it is necessary to divide the eigenvalue spectrum into three regions:

Region 1: $\eta \in [-1/\sigma, 1/\sigma]$,
Region 2: $\eta \in [-1, -1/\sigma]$ and $[1/\sigma, 1]$,
Region 3: $\eta \notin [-1, 1]$.

In Region 1 we find a two-fold degeneracy; i.e. we have two linearly independent eigenvectors for each eigenvalue. These can be written in the forms

$$\mathbf{F}_1^{(1)}(\eta, \mu) = \begin{bmatrix} -c_{12}\,\delta(\sigma\eta - \mu) \\ \dfrac{C\eta P}{\eta - \mu} + \delta(\eta - \mu)(c_{11} - 2\eta C T(\eta)) \end{bmatrix}, \tag{6a}$$

and

$$\mathbf{F}_2^{(1)}(\eta, \mu) = \begin{bmatrix} \dfrac{C\eta P}{\sigma\eta - \mu} + \delta(\sigma\eta - \mu)(c_{22} - 2\eta C T(\sigma\eta)) \\ -c_{21}\,\delta(\eta - \mu) \end{bmatrix}. \tag{6b}$$

In Region 2 there is only one solution,

$$\mathbf{F}^{(2)}(\eta, \mu) = \begin{bmatrix} \dfrac{c_{12}\eta}{\sigma\eta - \mu} \\ \dfrac{\eta f(\eta) P}{\eta - \mu} + \lambda(\eta)\,\delta(\eta - \mu) \end{bmatrix}, \tag{7}$$

where

$$f(\eta) = c_{22} - 2\eta C T(1/\sigma\eta), \tag{8}$$

and

$$\lambda(\eta) = 1 - 2\eta c_{22} T(\eta) - 2\eta c_{11} T(1/\sigma\eta) + 4\eta^2 C T(\eta) T(1/\sigma\eta). \tag{9}$$

Throughout this work the symbol P indicates that the Cauchy principal value is to be taken when integrals involving these functions are performed. Also, we have used the abbreviations C for det $\mathbf{C}$ and $T(x)$ for $\tanh^{-1}(x)$.

The discrete solutions (Region 3) are

$$\mathbf{F}_{i\pm}(\mu) = \begin{bmatrix} \dfrac{c_{12}\eta_i}{\sigma\eta_i \mp \mu} \\ \dfrac{c_{22} - 2C\eta_i T(1/\sigma\eta_i)}{\eta_i \mp \mu}\,\eta_i \end{bmatrix}, \tag{10}$$

where the η_i are the positive roots of the dispersion function

$$\Omega(z) = 1 - 2zc_{11}T(1/\sigma z) - 2zc_{22}T(1/z) + 4z^2CT(1/z)T(1/\sigma z). \tag{11}$$

There may be two or four roots of $\Omega(z)$ depending upon σ and the magnitudes of the c_{ij}. In the Appendix various possibilities for either two or four roots are discussed.

3. FULL-RANGE COMPLETENESS THEOREM

THEOREM I: *The set of functions* $\mathbf{F}_{i\pm}(\mu)$, $\mathbf{F}_1^{(1)}(\eta, \mu)$, $\mathbf{F}_2^{(1)}(\eta, \mu)$ *and* $\mathbf{F}^{(2)}(\eta, \mu)$ *for all* η *is complete for functions defined such that* $\mu \in [-1, 1]$, *in the sense that an arbitrary function,* $\mathbf{\Psi}(\mu)$, *defined for* $-1 \leq \mu \leq 1$ *can be expanded in the form*

$$\mathbf{\Psi}(\mu) = \sum_i A_{i+}\mathbf{F}_{i+}(\mu) + \sum_i A_{i-}\mathbf{F}_{i-}(\mu) + \int_{①} \alpha(\eta)\mathbf{F}_1^{(1)}(\eta, \mu)\,\mathrm{d}\eta$$
$$+ \int_{①} \beta(\eta)\mathbf{F}_2^{(1)}(\eta, \mu)\,\mathrm{d}\eta + \int_{②} \varepsilon(\eta)\mathbf{F}^{(2)}(\eta, \mu)\,\mathrm{d}\eta, \qquad -1 \leq \mu \leq 1. \tag{12}$$

Here the ranges of integration ① and ② refer to Regions 1 and 2 respectively.

Equation (12) can be considered a singular integral equation for the expansion coefficients. To prove the theorem, it is sufficient to prove that a solution exists. This, in turn, is done by solving the equation using the methods of MUSKHELISHVILI (1953). This yields expressions for the expansion coefficients. However, they are more conveniently obtained from the orthogonality relations developed in Section 4.

We begin by attempting an expansion in terms of the continuum modes alone, i.e.

$$\mathbf{\Psi}'(\mu) = \int_{①} \alpha(\eta)\mathbf{F}_1^{(1)}(\eta, \mu)\,\mathrm{d}\eta + \int_{①} \beta(\eta)\mathbf{F}_2^{(1)}(\eta, \mu)\,\mathrm{d}\eta + \int_{②} \varepsilon(\eta)\mathbf{F}^{(2)}(\eta, \mu)\,\mathrm{d}\eta,$$
$$\mu \in [-1, 1]. \tag{13}$$

Substituting the explicit forms of the eigensolutions, we obtain the two equations

$$\Psi_1'(\mu) = -\frac{c_{12}}{\sigma}\,\alpha(\mu/\sigma) + C\int_{①} \frac{\beta(\eta)\eta P}{\sigma\eta - \mu}\,\mathrm{d}\eta + \frac{\beta(\mu/\sigma)}{\sigma}\left[c_{22} - \frac{2\mu C}{\sigma}\,T(\mu)\right]$$
$$+ c_{12}\int_{②} \varepsilon(\eta)\,\frac{\eta}{\sigma\eta - \mu}\,\mathrm{d}\eta, \tag{14a}$$

and

$$\Psi_2'(\mu) = C \int_{①} \alpha(\eta) \frac{\eta P}{\eta - \mu} \, d\eta + \Theta_1(\mu)[c_{11} - 2\mu C T(\mu)]\alpha(\mu) - c_{21}\Theta_1(\mu)\beta(\mu)$$
$$+ \Theta_2(\mu)\lambda(\mu)\varepsilon(\mu) + \int_{②} \frac{\eta f(\eta) P}{\eta - \mu} \varepsilon(\eta) \, d\eta. \quad (14b)$$

Here $\Psi_1'(\mu)$ and $\Psi_2'(\mu)$ are the two components of $\boldsymbol{\Psi}'(\mu)$ and

$$\Theta_i(\mu) = 1, \qquad \mu \in \text{region } i,$$
$$= 0, \qquad \text{otherwise.}$$

If, in equation (14a), we make the change of variable $\mu/\sigma = \eta$, solve for α and substitute into equation (14b), we obtain

$$h(\mu) = \kappa(\mu)\omega(\mu) + \left\{\frac{\Omega^+(\mu) - \Omega^-(\mu)}{2\pi i \mu}\right\} \int_{-1}^{1} \frac{\eta P}{\eta - \mu} \kappa(\eta) \, d\eta, \qquad \mu \in [-1, 1], \quad (15)$$

where

$$h(\mu) \triangleq \Psi_2'(\mu) + \Theta_1(\mu) \frac{\sigma}{c_{12}} [c_{11} - 2\mu C T(\mu)]\Psi_1'(\sigma\mu) + \frac{C\sigma}{c_{12}} \int_{①} \frac{\eta P}{\eta - \mu} \Psi_1'(\sigma\eta) \, d\eta \quad (16)$$

and

$$\kappa(\eta) \triangleq \frac{C}{c_{12}} \beta(\eta)\Theta_1(\eta) + \varepsilon(\eta)\Theta_2(\eta). \quad (17)$$

Also, we have made use of the boundary values of the dispersion function, $\Omega(z)$, i.e.

$$\Omega^\pm(\eta) = 1 - 2\eta c_{11} T(\sigma\eta) - 2\eta c_{22} T(\eta) + C\eta^2[4T(\eta)T(\sigma\eta) - \pi^2]$$
$$\pm \pi i[(c_{11} + c_{22})\eta - 2C\eta^2(T(\eta) + T(\sigma\eta))], \qquad \eta \in \text{region } 1; \quad (18a)$$

$$\Omega^\pm(\eta) = 1 - 2\eta c_{11} T\left(\frac{1}{\sigma\eta}\right) - 2\eta c_{22} T(\eta) + C\eta^2\left[4T(\eta)T\left(\frac{1}{\sigma\eta}\right)\right]$$
$$\pm \pi i \left[c_{22}\eta - 2C\eta^2 T\left(\frac{1}{\sigma\eta}\right)\right], \qquad \eta \in \text{region } 2. \quad (18b)$$

The function $\omega(\mu)$ in equation (15) is defined by

$$\omega(\mu) \triangleq \frac{\Omega^+(\mu) + \Omega^-(\mu)}{2}. \quad (19)$$

By introducing the auxiliary function

$$N(z) \triangleq \frac{1}{2\pi i} \int_{-1}^{1} \frac{\eta\kappa(\eta) \, d\eta}{\eta - z} \quad (20)$$

with boundary values

$$N^\pm(\mu) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\eta P}{\eta - \mu} \kappa(\eta) \, d\eta \pm \tfrac{1}{2}\mu\kappa(\mu), \quad (21)$$

we can write equation (15) in the usual factored form (CASE, 1960). Thus

$$\mu h(\mu) = N^+(\mu)\Omega^+(\mu) - N^-(\mu)\Omega^-(\mu). \quad (22)$$

Since $h(\mu)$ is a known function (depending only on the expansion function $\mathbf{\Psi}'(\mu)$), we write the solution to equation (22) as

$$N(z) = \frac{1}{2\pi i \Omega(z)} \int_{-1}^{1} \mu h(\mu) \frac{\mathrm{d}\mu}{\mu - z}. \tag{23}$$

Noting from equation (20) that $N(z)$ is to be analytic in the complex plane cut from -1 to 1, we observe that equation (23) for $N(z)$ will have the correct analytic properties only if

$$\int_{-1}^{1} \mu h(\mu) \frac{\mathrm{d}\mu}{\mu \mp \eta_i} = 0. \tag{24}$$

This condition insures that the singularities introduced into $N(z)$ by the zeros of $\Omega(z)$ are removed. Recalling from equation (13) that the discrete terms in the expansion have not yet been included; we note that in actual fact,

$$\mathbf{\Psi}'(\mu) = \mathbf{\Psi}(\mu) - \sum_i A_{i+}\mathbf{F}_{i+}(\mu) - \sum_i A_{i-}\mathbf{F}_{i-}(\mu), \tag{25}$$

where $\mathbf{\Psi}(\mu)$ is an arbitrary function. Substituting equation (25) into equation (24) determines the coefficients $A_{i\pm}$. Since $N(z)$ is now known, $\kappa(\eta)$, and thus $\beta(\eta)$ and $\varepsilon(\eta)$, can be determined from its boundary values. This leaves only $\alpha(\eta)$ to be found and this coefficient is given by equation (14a).

The theorem is now proved; and we could, in fact, determine all of the expansion coefficients from this proof. It is, however, somewhat simpler to obtain them from the orthogonality relations that are developed in the next section.

4. ORTHOGONALITY AND NORMALIZATION

THEOREM II: *The functions* $\mathbf{F}_i^{(1)}(\eta, \mu)$, $\mathbf{F}^{(2)}(\eta, \mu)$ *and* $\mathbf{F}_{i\pm}(\mu)$ *are orthogonal to the corresponding solutions of the adjoint equation on the range* $[-1, 1]$ *with weight function* μ. *That is*

$$\int_{-1}^{1} \mu \tilde{\mathbf{F}}^{\dagger}(\eta, \mu)\mathbf{F}(\eta', \mu)\,\mathrm{d}\mu = 0, \qquad \eta \neq \eta'. \tag{26}$$

Here the superscript tilde denotes the transpose operation. We rewrite equation (1) and the adjoint equation below in symbolic form:

$$\left(\frac{\mathbf{\Sigma}}{\mu} - \frac{\mathbf{C}}{\mu}\int_{-1}^{1} .\,\mathrm{d}\mu\right)\mathbf{F}(\eta, \mu) = \frac{1}{\eta}\,\mathbf{F}(\eta, \mu) \tag{27a}$$

$$\left(\frac{\mathbf{\Sigma}}{\mu} - \frac{\tilde{\mathbf{C}}}{\mu}\int_{-1} .\,\mathrm{d}\mu\right)\mathbf{F}^{\dagger}(\eta', \mu) = \frac{1}{\eta'}\,\mathbf{F}^{\dagger}(\eta', \mu). \tag{27b}$$

It is easily verified that equations (27) have identical eigenvalue spectra. Also, we note that

$$\mathbf{F}^{\dagger}(\eta, \mu, \mathbf{C}) = \mathbf{F}(\eta, \mu, \tilde{\mathbf{C}}), \tag{28}$$

i.e. the adjoint solutions are found by simply changing c_{ij} to c_{ji} in the corresponding ordinary solutions.

To prove the theorem, multiply equation (27a) from the left by $\mu\tilde{\mathbf{F}}^{\dagger}(\eta', \mu)$. Then multiply the *transpose* of equation (27b) from the right by $\mu\mathbf{F}(\eta, \mu)$. Integrating both equations thus formed over μ from -1 to 1 and subtracting, we get

$$\left(\frac{1}{\eta} - \frac{1}{\eta'}\right) \int_{-1}^{1} \mu \tilde{\mathbf{F}}^{\dagger}(\eta', \mu)\mathbf{F}(\eta, \mu)\, \mathrm{d}\mu = 0. \tag{29}$$

This proves the theorem.

In order to use the orthogonality theorem to calculate the various expansion coefficients, we must first determine the necessary normalization integrals. Of course, we must not overlook the fact that there are two degenerate eigensolutions in region 1. These can be orthogonalized by using a Schmidt-type procedure. Firstly we present the normalization integrals. Defining

$$(\mathbf{F}, \mathbf{G}) \triangleq \int_{-1}^{1} \mu \tilde{\mathbf{F}}^{\dagger}(\eta', \mu)\mathbf{G}(\eta, \mu)\, \mathrm{d}\mu, \tag{30}$$

we have

$$(\mathbf{F}_{i\pm}, \mathbf{F}_{i\pm}) = N_{i\pm}, \tag{31a}$$

$$(\mathbf{F}_i^{(1)}, \mathbf{F}_j^{(1)}) = N_{ij}\, \delta(\eta - \eta'), \tag{31b}$$

and

$$(\mathbf{F}^{(2)}, \mathbf{F}^{(2)}) = N_2\, \delta(\eta - \eta'). \tag{31c}$$

Here

$$N_{i\pm} = \pm 2\eta_i^2 \left\{ c_{12}c_{21}\left[\frac{\sigma\eta_i}{(\sigma\eta_i)^2 - 1} - T(1/\sigma\eta_i)\right] + [c_{22} - 2C\eta_i T(1/\sigma\eta_i)]^2 \left[\frac{\eta_i}{\eta_i^2 - 1} - T(1/\eta_i)\right]\right\}, \tag{32a}$$

$$N_{ij} = -c_{ji}\eta\{c_{11} + c_{22} - 2\eta C[T(\eta) + T(\sigma\eta)]\}, \tag{32b}$$

$$N_{11} = \eta\{c_{12}c_{21} + [c_{11} - 2\eta CT(\eta)]^2 + \pi^2 C^2 \eta^2\}, \tag{32c}$$

$$N_{22} = \eta\{c_{12}c_{21} + [c_{22} - 2\eta CT(\sigma\eta)]^2 + \pi^2 C^2 \eta^2\}, \tag{32d}$$

and

$$N_2 = \eta\Omega_2^{+}(\eta)\Omega_2^{-}(\eta). \tag{32e}$$

The subscript on $\Omega_i^{\pm}(\eta)$ denotes that η is contained in region 'i' [cf. equations (18)]. Explicitly, we write

$$\Omega_1^{+}(\eta)\Omega_1^{-}(\eta) = \{1 - 2\eta c_{11}T(\sigma\eta) - 2\eta c_{22}T(\eta) + \eta^2 C[4T(\eta)T(\sigma\eta) - \pi^2]\}^2 + \pi^2\eta^2\{2C\eta[T(\eta) + T(\sigma\eta)] - c_{11} - c_{22}\}^2 \tag{33a}$$

and

$$\Omega_2^{+}(\eta)\Omega_2^{-}(\eta) = \{1 - 2\eta c_{22}T(\eta) - 2\eta c_{11}T(1/\sigma\eta) + 4\eta^2 CT(\eta)T(1/\sigma\eta)\}^2 + \pi^2\eta^2\{c_{22} - 2\eta CT(1/\sigma\eta)\}^2. \tag{33b}$$

Since $\mathbf{F}_1^{(1)}$ and $\mathbf{F}_2^{(1)}$ are not orthogonal for $\eta = \eta'$, we introduce two new functions, $\mathbf{X}_1(\eta, \mu)$ and $\mathbf{X}_2(\eta, \mu)$, such that $\mathbf{X}_1$ is orthogonal to $\mathbf{F}_2^{(1)}$ and $\mathbf{X}_2$ is orthogonal to $\mathbf{F}_1^{(1)}$ (clearly both $\mathbf{X}_i$'s are orthogonal to $\mathbf{F}^{(2)}$ and the $\mathbf{F}_{i\pm}$'s). Therefore

$$\mathbf{X}_1(\eta, \mu) \triangleq N_{22}\mathbf{F}_1^{(1)}(\eta, \mu) - N_{12}\mathbf{F}_2^{(1)}(\eta, \mu) \tag{34a}$$

and

$$\mathbf{X}_2(\eta, \mu) \triangleq N_{11}\mathbf{F}_2^{(1)}(\eta, \mu) - N_{21}\mathbf{F}_1^{(1)}(\eta, \mu). \tag{34b}$$

It follows that

$$(\mathbf{X}_1, \mathbf{F}_1^{(1)}) = N_1\,\delta(\eta - \eta'), \tag{35a}$$

$$(\mathbf{X}_1, \mathbf{F}_2^{(1)}) = 0, \tag{35b}$$

$$(\mathbf{X}_2, \mathbf{F}_2^{(1)}) = N_1\,\delta(\eta - \eta'), \tag{35c}$$

and

$$(\mathbf{X}_2, \mathbf{F}_1^{(1)}) = 0, \tag{35d}$$

where*

$$N_1 = \eta^2 C^2 \Omega_1^{+}(\eta)\Omega_1^{-}(\eta). \tag{36}$$

With the formalism developed in this section we are now able to solve for the infinite-medium Green's function.

5. THE INFINITE-MEDIUM GREEN'S FUNCTION

We wish to construct a solution to the homogeneous two-group transport equation which satisfies the 'jump' condition at the source location and which also vanishes at $x = \pm\infty$. Clearly this is possible only for some values of the c_{ij} (see the Appendix for a discussion of this point) since we are restricted to a non-multiplying medium. For complete generality we need to find two Green's functions, $\mathbf{G}_1$ and $\mathbf{G}_2$, corresponding to the two sources

$$\mathbf{q}_1 = \delta(x)\,\delta(\mu - \mu_1)\begin{bmatrix}1\\0\end{bmatrix}, \tag{37a}$$

and

$$\mathbf{q}_2 = \delta(x)\,\delta(\mu - \mu_2)\begin{bmatrix}0\\1\end{bmatrix} \tag{37b}$$

respectively. The superposition principle can then be used to construct the desired solution for any arbitrary source distribution. Since the solutions for the two $\mathbf{G}_i$'s are similar, we thus discuss only $\mathbf{G}_1$ in detail. The solutions to equation (1) that vanish appropriately are

$$\begin{aligned}\mathbf{G}_1(0 \to x, \mu_1 \to \mu) = {} & \sum_i A_{i+}\mathrm{e}^{-x/\eta_i}\mathbf{F}_{i+}(\mu) + \int_0^{1/\sigma} \alpha(\eta)\mathrm{e}^{-x/\eta}\mathbf{F}_1^{(1)}(\eta, \mu)\,\mathrm{d}\eta \\ & + \int_0^{1/\sigma} \beta(\eta)\mathrm{e}^{-x/\eta}\mathbf{F}_2^{(1)}(\eta, \mu)\,\mathrm{d}\eta \\ & + \int_{1/\sigma}^{1} \varepsilon(\eta)\mathrm{e}^{-x/\eta}\mathbf{F}^{(2)}(\eta, \mu)\,\mathrm{d}\eta, \qquad x > 0\end{aligned} \tag{38a}$$

and

$$\begin{aligned}\mathbf{G}_1(0 \to x, \mu_1 \to \mu) = {} & -\sum_i A_{i-}\mathrm{e}^{x/\eta_i}\mathbf{F}_{i-}(\mu) - \int_{-1/\sigma}^{0} \alpha(\eta)\mathrm{e}^{-x/\eta}\mathbf{F}_1^{(1)}(\eta, \mu)\,\mathrm{d}\eta \\ & - \int_{-1/\sigma}^{0} \beta(\eta)\mathrm{e}^{-x/\eta}\mathbf{F}_2^{(1)}(\eta, \mu)\,\mathrm{d}\eta \\ & - \int_{-1}^{-1/\sigma} \varepsilon(\eta)\mathrm{e}^{-x/\eta}\mathbf{F}^{(2)}(\eta, \mu)\,\mathrm{d}\eta, \qquad x < 0.\end{aligned} \tag{38b}$$

* The limit $C = 0$ is not easily deduced from the formalism of this paper. This is evident at once since the degenerate eigensolutions $\mathbf{F}_i^{(1)}(\eta, \mu)$ in equations (6) are not linearly independent in the limit of C tending to zero. The case of $C = 0$ is presented in detail by BARAN (1966, private communication).

Applying the 'jump' boundary condition,

$$\mathbf{G}_1(0 \to 0^+, \mu_1 \to \mu) - \mathbf{G}_1(0 \to 0^-, \mu_1 \to \mu) = \frac{\delta(\mu - \mu_1)}{\mu} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{39}$$

we obtain

$$\frac{\delta(\mu - \mu_1)}{\mu} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sum_i A_{i+}\mathbf{F}_{i+}(\mu) + \sum_i A_{i-}\mathbf{F}_{i-}(\mu) + \int_{①} \alpha(\eta)\mathbf{F}_1^{(1)}(\eta, \mu)\, d\eta$$
$$+ \int_{①} \beta(\eta)\mathbf{F}_2^{(1)}(\eta, \mu)\, d\eta + \int_{②} \varepsilon(\eta)\mathbf{F}^{(2)}(\eta, \mu)\, d\eta, \qquad \mu, \mu_1 \in [-1, 1]. \tag{40}$$

This is simply a full-range expansion of the function

$$\frac{\delta(\mu - \mu_1)}{\mu} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{41}$$

The expansion coefficients $A_{i\pm}$, $\alpha(\eta)$, $\beta(\eta)$ and $\varepsilon(\eta)$ are thus obtained by applying the orthogonality relations to equation (40). The coefficients $A_{i\pm}$, for example, are found by taking the scalar product of $\mathbf{F}_{i\pm}(\mu)$ and equation (40). It follows that

$$A_{i\pm} = \frac{1}{N_{i\pm}} \int_{-1}^{1} \delta(\mu - \mu_1)\tilde{\mathbf{F}}_{i\pm}^{\dagger}(\mu) \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\mu, \tag{42a}$$

i.e.

$$A_{i\pm} = \frac{1}{N_{i\pm}} \tilde{\mathbf{F}}_{i\pm}^{\dagger}(\mu_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{42b}$$

Multiplying out equation (42b), we see that $A_{i\pm}$ is given by

$$A_{i\pm} = \frac{c_{21}\eta_i}{\sigma\eta_i \mp \mu_1} \cdot \frac{1}{N_{i\pm}}, \tag{43}$$

where $N_{i\pm}$ is given by equation (32a). For the remaining coefficients we find

$$\alpha(\eta) = -\frac{1}{N_1(\eta)} \Big\{ N_{22}(\eta)c_{21}\,\delta(\sigma\eta - \mu_1)$$
$$+ N_{12}(\eta) \left[\frac{C\eta P}{\sigma\eta - \mu_1} + \delta(\sigma\eta - \mu_1)(c_{22} - 2\eta CT(\sigma\eta)) \right] \Big\}, \tag{44a}$$

$$\beta(\eta) = +\frac{1}{N_1(\eta)} \Big\{ N_{21}(\eta)c_{21}\,\delta(\sigma\eta - \mu_1)$$
$$+ N_{11}(\eta) \left[\frac{C\eta P}{\sigma\eta - \mu_1} + \delta(\sigma\eta - \mu_1)(c_{22} - 2\eta CT(\sigma\eta)) \right] \Big\} \tag{44b}$$

and

$$\varepsilon(\eta) = \frac{1}{N_2(\eta)} \cdot \frac{c_{21}\eta}{\sigma\eta - \mu_1}. \tag{44c}$$

All of the expansion coefficients are now determined. The solution for the source $\mathbf{q}_1$ is therefore known and is given by equations (38), (43) and (44). In a similar manner we find the Green's function $\mathbf{G}_2(0 \to x, \mu_2 \to \mu)$ to be

$$\mathbf{G}_2(0 \to x, \mu_2 \to \mu) = \sum_i B_{i+}\, e^{-x/\eta_i}\mathbf{F}_{i+}(\mu) + \int_0^{1/\sigma} a(\eta)\, e^{-x/\eta}\mathbf{F}_1^{(1)}(\eta, \mu)\, d\eta$$
$$+ \int_0^{1/\sigma} b(\eta)\, e^{-x/\eta}\mathbf{F}_2^{(1)}(\eta, \mu)\, d\eta$$
$$+ \int_{1/\sigma}^{1} d(\eta)\, e^{-x/\eta}\mathbf{F}^{(2)}(\eta, \mu)\, d\eta, \qquad x > 0 \tag{45a}$$

and

$$\mathbf{G}_2(0 \to x, \mu_2 \to \mu) = -\sum_i B_{i-}\, e^{x/\eta_i}\mathbf{F}_{i-}(\mu) - \int_{-1/\sigma}^{0} a(\eta)\, e^{-x/\eta}\mathbf{F}_1^{(1)}(\eta, \mu)\, d\eta$$
$$- \int_{-1/\sigma}^{0} b(\eta)\, e^{-x/\eta}\mathbf{F}_2^{(1)}(\eta, \mu)\, d\eta$$
$$- \int_{-1}^{-1/\sigma} d(\eta)\, e^{-x/\eta}\mathbf{F}^{(2)}(\eta, \mu)\, d\eta, \qquad x < 0 \tag{45b}$$

where

$$B_{i\pm} = \frac{\mu_2}{N_{i\pm}} \cdot \frac{c_{22} - 2C\eta_i T(1/\sigma\eta_i)}{\eta_i \mp \mu_2}\eta_i, \tag{46a}$$

$$a(\eta) = +\frac{1}{N_1(\eta)}\left\{N_{12}c_{12}\,\delta(\eta - \mu_2) + N_{22}\left[\frac{C\eta P}{\eta - \mu_2} + \delta(\eta - \mu_2)(c_{11} - 2\eta CT(\eta))\right]\right\}, \tag{46b}$$

$$b(\eta) = -\frac{1}{N_1(\eta)}\left\{N_{11}c_{12}\,\delta(\eta - \mu_2) + N_{21}\left[\frac{C\eta P}{\eta - \mu_2} + \delta(\eta - \mu_2)(c_{11} - 2\eta CT(\eta))\right]\right\} \tag{46c}$$

and

$$d(\eta) = \frac{1}{N_2(\eta)}\left\{\frac{\eta f(\eta) P}{\eta - \mu_2} + \delta(\eta - \mu_2)\lambda(\eta)\right\}. \tag{46d}$$

Now that the most general Green's functions are known, such quantities as $\boldsymbol{\rho}_i(0 \to x, \mu_i)$ and $\mathbf{G}_i(0 \to x, \mu)$, the total flux from a directed source and the Green's functions for an isotropic source, are easily obtained by integration over the various directions. We omit these operations as they are completely analogous to the steps taken in one-speed theory by CASE (1961).

Acknowledgments—The authors would like to express their gratitude to Professor R. L. MURRAY for his constructive comments during the preparation of this manuscript. We would also like to thank Mr. W. BARAN for communicating to us some of his results prior to publication. In addition, one of the authors (C. E. S.) wishes to acknowledge the many erudite suggestions made by Professor P. F. ZWEIFEL.

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APPENDIX

The discrete eigenvalues, $\pm\eta_i$, are the zeros of the dispersion function

$$\Omega(z) = 1 - 2c_{11}zT(1/\sigma z) - 2c_{22}zT(1/z) + 4Cz^2T(1/z)T(1/\sigma z). \tag{47}$$

By calculating the change in the argument of the function $\Omega(z)$ around a contour that includes all of the z-plane save the segment of the real line $[-1, 1]$, one is able to determine the exact number of zeros of $\Omega(z)$. We have made this determination and the results are tabulated in Table 1. In addition, we have investigated the nature of these roots. Our results are in agreement with Baran, who also considered this problem.

In order for the infinite-medium Green's function derived in Section 5 to have meaning, we must postulate that the medium is neither conservative nor multiplying. By restricting ourselves to the cases where there are only finite, real, discrete eigenvalues, the Green's functions already developed are correct. The necessary information is available in Table 1.

Table 1.—The zeros of the dispersion function

Conditions			Roots
$C = 0$		$c_{11} + \sigma c_{22} < \sigma/2$	2 Real
		$c_{11} + \sigma c_{22} > \sigma/2$	2 Imaginary
		$c_{11} + \sigma c_{22} = \sigma/2$	2 Infinite
$C < 0$		$c_{11} + \sigma c_{22} - 2C < \sigma/2$	2 Real
		$c_{11} + \sigma c_{22} - 2C > \sigma/2$	2 Imaginary
		$c_{11} + \sigma c_{22} - 2C = \sigma/2$	2 Infinite
$C > 0$, $c_{22} > 2CT(1/\sigma)$		$c_{11} + \sigma c_{22} - 2C < \sigma/2$	2 Real
		$c_{11} + \sigma c_{22} - 2C > \sigma/2$	2 Imaginary
		$c_{11} + \sigma c_{22} - 2C = \sigma/2$	2 Infinite
$C > 0$, $c_{22} \leq 2CT(1/\sigma)$	$c_{11} \leq \sigma/2$ and $c_{22} \geq 1/2$ or $c_{11} \geq \sigma/2$ and $c_{22} \leq 1/2$		2 Real and 2 Imaginary
	$c_{11} < \sigma/2$ and $c_{22} < 1/2$	$c_{11} + \sigma c_{22} - 2C < \sigma/2$	4 Real
		$c_{11} + \sigma c_{22} - 2C > \sigma/2$	2 Real and 2 Imaginary
		$c_{11} + \sigma c_{22} - 2C = \sigma/2$	2 Real and 2 Infinite
	$c_{11} > \sigma/2$ and $c_{22} > 1/2$	$c_{11} + \sigma c_{22} - 2C < \sigma/2$	4 Imaginary
		$c_{11} + \sigma c_{22} - 2C > \sigma/2$	2 Real and 2 Imaginary
		$c_{11} + \sigma c_{22} - 2C = \sigma/2$	2 Imaginary and 2 Infinite