An Asymptotic Solution in the Theory of Neutron Moderation

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ABSTRACT
An asymptotic solution to the equation that describes the slowing down of neutrons is given.

ANALYSIS
The basic equation in the elementary theory of neutron moderation can be written as

$$ \Phi(u) = \delta(u) + c \int_{u-q}^{u} \Phi(u') K(u - u') du' $$

where

$$ \Phi(u) = \text{flux} $$

$$ \delta(u) = \text{Dirac's delta functional} $$

$$ c = \frac{\Sigma_e}{\Sigma} $$

$$ q = -\ln \alpha $$

$$ K(u) = \frac{e^{-u}}{1 - \alpha} \quad 0 \leq u \leq q $$

$$ K(u) = 0, \quad \text{otherwise.} $$

Following the work of Marshak,1 Ferziger and Zweifel2 and Williams3 have expressed the asymptotic solution to Eq. (1) as

$$ \Phi_{asy}(u) = \frac{c \left[ 1 - \exp[-q(s + 1)] \right] \exp(s_u u)}{1 - \alpha - cq \exp[-q(s + 1)]} \quad (2) $$

where $s_u$ is the appropriate real solution $\epsilon(-\infty, 0)$ of

$$ (1 - \alpha)(s + 1) = c \left[ 1 - \exp[-q(s + 1)] \right] \quad (3) $$

Equation (3) has only two real solutions, one of which clearly is $s = -1$; we note, however, that $s_u \neq -1$, except for the very special case for which $cq = 1 - \alpha$.

In this Note we wish to report an exact expression for $s_u$, and thus we let $\beta = cq/(\alpha - 1)$ and $\lambda = \beta + q(s + 1)$ so that Eq. (3) can be written as

$$ \lambda e^\lambda = \beta e^\beta \quad (4) $$

The work of Siewert and Burniston4 can now be used to solve Eq. (4) for the real solution $\lambda \neq \beta$ and thus to establish an explicit expression for $s_u$:

$$ s_u = -1 - \frac{(\beta + 1)}{q} \exp \left\{ -\frac{1}{\pi} \int_{0}^{\infty} \left[ f(x) - \pi \right] \frac{dx}{x - \beta} \right\} \quad (5) $$

where

$$ f(x) = \tan^{-1} \left( \frac{\pi}{\beta + \ln |\beta| - x - \ln x} \right) \quad (6) $$

is continuous and such that $f(0) = 0$. The solution given by Eq. (5) has been evaluated numerically to yield results accurate to five significant figures.

To report some numerical results, we write Eq. (5) as

$$ s_u = -1 - \frac{(\beta + 1)}{q} K(\beta) \quad (7) $$

and list in the accompanying table the function $K(\beta)$.

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