Int. J. Engng Sci., 1974, Vol. 12, pp. 861–863. Pergamon Press. Printed in Great Britain

AN EXACT ANALYTICAL SOLUTION FOR THE POSITION–TIME RELATIONSHIP FOR AN INVERSE-DISTANCE-SQUARED FORCE

C. E. SIEWERT and W. O. DOGGETT

Departments of Nuclear Engineering and Physics, North Carolina State University, Raleigh, North Carolina 27607 U.S.A.

(*Received* 5 *September* 1973)

Abstract—The theory of complex variables is used to solve analytically the transcendental equation which defines the position, as a function of time, in an inverse-distance-squared force field, and thereby the desired solution is reduced to elementary quadrature.

1. INTRODUCTION

THE MOTION of a nonrelativistic particle with charge q, mass m, and no angular momentum in an inverse-distance-squared force field, such as arises in the free expansion of a charged-particle bunch [1], is described (in rationalized m.k.s. units) by

$$m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{qQ}{4\pi\epsilon_0 r^2}, \tag{1}$$

where Q is the constant total charge spherically symmetrically distributed within a sphere of radius r. Multiplying equation (1) by $(\mathrm{d}r/\mathrm{d}t)\mathrm{d}t = \mathrm{d}r$ and integrating once, we obtain the energy integral

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{qQ}{2\pi\epsilon_0 m}\left(\frac{1}{r_0} - \frac{1}{r}\right), \tag{2}$$

with the initial conditions $\mathrm{d}r/\mathrm{d}t = 0$, $r = r_0$, $t = 0$. A second integral yields

$$r(t) = r_0 x(t), \tag{3}$$

in which $x(t)$ is given by the transcendental relation

$$\sqrt{[x(x-1)]} + \ln[\sqrt{(x)} + \sqrt{(x-1)}] = \frac{t}{\tau}, \tag{4}$$

where

$$\tau = (2\pi\epsilon_0 m r_0^3/qQ)^{1/2}.$$

In the following section the solution of equation (4) is established analytically and is thus reduced to quadrature.

2. ANALYSIS

We wish to use a recently reported method [2] to solve equation (4), and thus we first consider the function of a complex variable

$$\Lambda(z) = T - \sqrt{(z)}\sqrt{(z-1)} - \log[\sqrt{(z)} + \sqrt{(z-1)}], \tag{5}$$

where $T = t/\tau$ and the square-root and log functions are to be interpreted as the principal branches. It is clear that $\Lambda(z)$ is analytic in the complex plane cut from $-\infty$ to 1 along the real axis and that $\Lambda(z)/z \to -1$ as $|z| \to \infty$. In addition, the limiting values $\Lambda^{\pm}(x)$ of $\Lambda(z)$ as z approaches the branch cut from above (+) and below (−) can be deduced from equation (5):

$$\Lambda^{\pm}(x) = T + \sqrt{(x^2 - x)} - \ln[\sqrt{(-x)} + \sqrt{(1-x)}] \mp \frac{\pi}{2} i, \; x \in (-\infty, 0), \tag{6a}$$

and

$$\Lambda^{\pm}(x) = T \mp \left[\tan^{-1}\sqrt{\left(\frac{1}{x} - 1\right)} + \sqrt{(x - x^2)}\right] i, \; x \in (0, 1). \tag{6b}$$

We can now use the argument principle[3] to show that $\Lambda(z)$ has only one zero, say z_0, in the complex plane and thus the solution of equation (4) is simply $x(t) = z_0(t)$.

Since $\Lambda(z)$ has only one zero in the complex plane, it now follows that

$$\phi(z) = \frac{\Lambda(z)}{z_0 - z} \tag{7}$$

is a canonical solution of the Riemann problem[4]

$$\phi^+(x) = \frac{\Lambda^+(x)}{\Lambda^-(x)} \phi^-(x), \; x \in (-\infty, 1). \tag{8}$$

The methods discussed by Muskhelishvili[4] can now be used to solve the Riemann problem defined by equation (8); we find

$$\phi(z) = X_1(z) X_2(z), \tag{9}$$

$$X_1(z) = \exp\left[-\frac{1}{\pi}\int_0^1 \theta_1(x) \frac{\mathrm{d}x}{x(x-1) - z(x-1)^2}\right] \tag{10a}$$

and

$$X_2(z) = \exp\left[-\frac{1}{\pi}\int_0^1 \theta_2(x) \frac{\mathrm{d}x}{x - z}\right]. \tag{10b}$$

Here

$$\theta_1(x) = \tan^{-1}\left\{\frac{\pi/2}{T + \frac{\sqrt{x}}{1-x} - \ln\left[\sqrt{\left(\frac{x}{1-x}\right)} + \sqrt{\left(\frac{1}{1-x}\right)}\right]}\right\} \tag{11a}$$

and

$$\theta_2(x) = \tan^{-1}\left[\frac{\tan^{-1}\sqrt{\left(\frac{1}{x} - 1\right)} + \sqrt{(x - x^2)}}{T}\right]. \tag{11b}$$

We can now enter equation (9) into equation (7) to obtain an explicit result for z_0:

$$z_0 = z + \frac{\Lambda(z)}{X_1(z) X_2(z)}. \tag{12}$$

Equation (12) is, of course, an identity in the z plane, and thus we can assign z any convenient value. For example, on taking $z = 1$ in equation (12) we find that the desired solution of equation (4) can be written as

$$x(t) = 1 + \frac{t}{\tau} \exp\left[-\frac{1}{\pi}\int_0^1 \theta(y, t) \frac{\mathrm{d}y}{1-y}\right], \tag{13}$$

where $\theta(y,t)\epsilon[0,\pi]$ is given by

$$\theta(y,t)=\tan^{-1}\frac{N(y,t)}{D(y,t)}. \tag{14}$$

with

$$N(y,t)=\frac{\pi}{2}\frac{t}{\tau}+\left\{\frac{t}{\tau}+\frac{\sqrt{(y)}}{1-y}-\ln\left[\sqrt{\left(\frac{y}{1-y}\right)}+\sqrt{\left(\frac{1}{1-y}\right)}\right]\right\}\left[\tan^{-1}\sqrt{\left(\frac{1}{y}-1\right)}+\sqrt{(y-y^2)}\right] \tag{15a}$$

and

$$D(y,t)=\frac{t}{\tau}\left\{\frac{t}{\tau}+\frac{\sqrt{(y)}}{1-y}-\ln\left[\sqrt{\left(\frac{y}{1-y}\right)}+\sqrt{\left(\frac{1}{1-y}\right)}\right]\right\}-\frac{\pi}{2}\left[\tan^{-1}\sqrt{\left(\frac{1}{y}-1\right)}+\sqrt{(y-y^2)}\right]. \tag{15b}$$

A Gaussian quadrature scheme has been used to evaluate our solution given by equation (13), and without excessive effort accuracy to six significant figures was achieved.

REFERENCES

[1] W. J. GALLAGHER, *I.E.E.E. Trans. Nucl. Sci.* **14**, 607 (1967).
[2] E. E. BURNISTON and C. E. SIEWERT, *Proc. Camb. Phil. Soc.* **73**, 111 (1973).
[3] L. V. AHLFORS, *Complex Analysis*, McGraw-Hill, (1953).
[4] N. I. MUSKHELISHVILI, *Singular Integral Equations*. Noordhoff (1953).

(*Received 5 September* 1973)

Résumé—La théorie des variables complexes est utilisée pour résoudre analytiquement l'équation transcendante qui définit une position, comme fonction du temps, dans un champ de force inverse du carré de la distance, et ainsi la solution recherchée est ramenée à une quadrature élémentaire.

Zusammenfassung—Es wird die Theorie komplexer Veränderlicher verwendet, um analytisch die transzendentale Gleichung zu lösen, die die Lage, als Funktion der Zeit, in einem umkehrdistanzquadrierten Kraftfeld festlegt, wodurch die gewünschte Lösung auf eine elementare Quadratur reduziert wird.

Sommario—La teoria delle variabili complesse viene usata per risolvere analiticamente l'equazione trascendentale che definisce in funzione del tempo la posizione di un campo di forza proporzionale al reciproco del quadrato della distanza, e perciò la soluzione desiderata viene ridotta ad una semplice quadratura.

Абстракт — На основе теории комплексных переменных дано аналитическое решение трансцендентного уравнения, определяющего положение в зависимости от времени в поле силы, возвышенной в квадрат обратного расстояния, что приводит заданное решение к форме элементарной квадратуры.