FURTHER RESULTS CONCERNING EXACT ANALYTICAL SOLUTIONS BASIC TO TWO-BODY ORBITS

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Abstract. Complex-variable techniques are used to establish exact analytical solutions to a class of two-body problems. In view of Lambert's theorem, two points on the conic, the chord-distance between the two points, and the time interval are considered given, and subsequently the solutions for the semi-major axis required to define the orbit are developed and expressed ultimately in terms of elementary quadratures.

1. Introduction

In a recent paper (Burniston and Siewert, 1973a) hereafter referred to as I, we applied a technique (Burniston and Siewert, 1973b) which makes use of Riemann problems in the theory of analytic functions to the problem of determining the semi-major axis of an orbit, given two points on the orbit, the chord-distance between the two points and the time interval. Specifically, with reference to Figure 1, we note (Wintner, 1947) that the points P_1 and P_2 are located at distances r_1 and r_2 respectively from the attracting focus S and c is the chord-distance. It is well known that the problem is not well posed unless the shaded region A in Figure 1 is known to contain or not to contain the attracting focus S. If the orbit is an ellipse we also need to know whether or not A contains the empty focus S^*. In I we presented solutions for the cases: (i) hyperbolic or parabolic orbits with $S \notin A$, (ii) elliptic orbits with $S \notin A$ *and* $S^* \notin A$ and (iii) elliptic orbits with both $S \in A$ and $S^* \in A$. For elliptic orbits, we also allowed the body to have made k specified revolutions in addition to the arc P_1P_2, with the condition that either

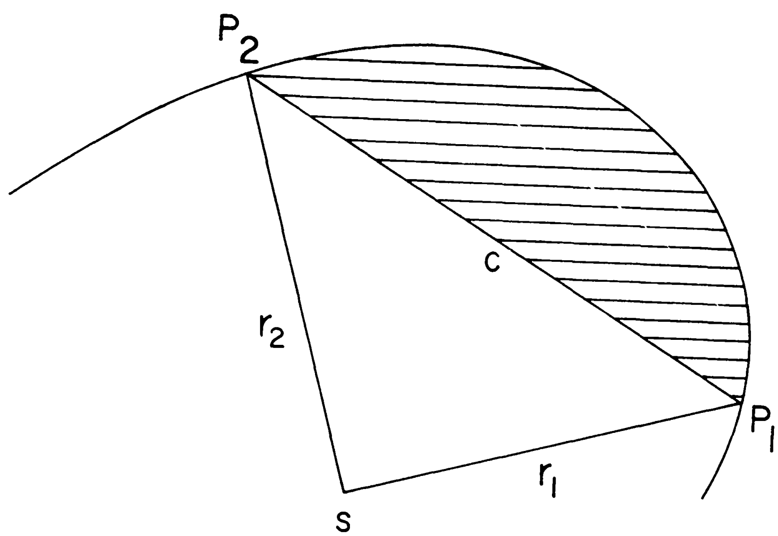

Fig. 1. The reference configuration.

$S \notin A$ and $S^* \notin A$ or $S \in A$ and $S^* \in A$. We now wish to report solutions for the remaining cases: (i) hyperbolic or parabolic orbits with $S \in A$, (ii) elliptic orbits with $S \in A$ and $S^* \notin A$ and (iii) elliptic orbits with $S \notin A$ and $S^* \in A$. We shall also include the possibility that the body has made k revolutions in addition to the arc P_1P_2, with either $S \in A$ *and* $S^* \notin A$ or $S \notin A$ *and* $S^* \in A$.

2. Preliminary Analysis

We first consider the hyperbolic and parabolic cases such that $S \in A$ and those elliptic orbits with $S \in A$ and $S^* \notin A$. As discussed by Wintner (1947), the problem here can be defined by assigning an appropriate path of integration to

$$M = \oint_{(r_1+r_2-c)/2}^{(r_1+r_2+c)/2} \left[\frac{1}{x} + h\right]^{-1/2} \mathrm{d}x, \tag{1}$$

where r_1, r_2 and c are the distances so denoted in Figure 1, and

$$M = \sqrt{2}(t_2 - t_1) \tag{2}$$

is the time interval. We consider that M, r_1, r_2 and c are given and thus seek the energy constant h, or alternatively, the semi-major axis

$$a = -\frac{1}{2h}. \tag{3}$$

We find, on letting $h = -z$, that we can change the integration variable in Equation (1) to obtain

$$M = \left(\int_\alpha^\infty + \int_\beta^\infty\right) \frac{1}{\sqrt{t - z}} \frac{\mathrm{d}t}{t^2}, \tag{4}$$

where

$$\alpha = 2(r_1 + r_2 + c)^{-1} \tag{5a}$$

and

$$\beta = 2(r_1 + r_2 - c)^{-1}. \tag{5b}$$

In addition, we note that the square-root function in Equation (4) is to be interpreted as the principal branch. It thus follows that the zeros, say z_i, of

$$\Omega(z) = M - \left(\int_\alpha^\infty + \int_\beta^\infty\right) \frac{1}{\sqrt{t - z}} \frac{\mathrm{d}t}{t^2} \tag{6}$$

will yield, by way of $h = -z_i$ or Equation (3), the desired solution(s). Viewed as a function of a complex variable, we observe that $\Omega(z)$ is analytic in the plane cut from α to ∞ along the real axis.

We require the limiting values of $\Omega(z)$ as z approaches the cut $[\alpha, \infty)$ from above

(+) and below (−), and thus from Equation (6) we deduce that

$$\Omega^{\pm}(x) = M - \left(\int_x^\beta + 2\int_\beta^\infty\right) \frac{1}{\sqrt{t-x}} \frac{\mathrm{d}t}{t^2} \mp i \int_\alpha^x \frac{1}{\sqrt{x-t}} \frac{\mathrm{d}t}{t^2}, \quad \alpha \leqslant x \leqslant \beta, \tag{7a}$$

and

$$\Omega^{\pm}(x) = M - 2\int_x^\infty \frac{1}{\sqrt{t-x}} \frac{\mathrm{d}t}{t^2} \mp i \left(\int_\alpha^\beta + 2\int_\beta^x\right) \frac{1}{\sqrt{x-t}} \frac{\mathrm{d}t}{t^2}, \quad x \geqslant \beta. \tag{7b}$$

The number of zeros of $\Omega(z)$ can now be computed by using the argument principle (Ahlfors, 1953) in a domain determined by the contour C given in Figure 2, with

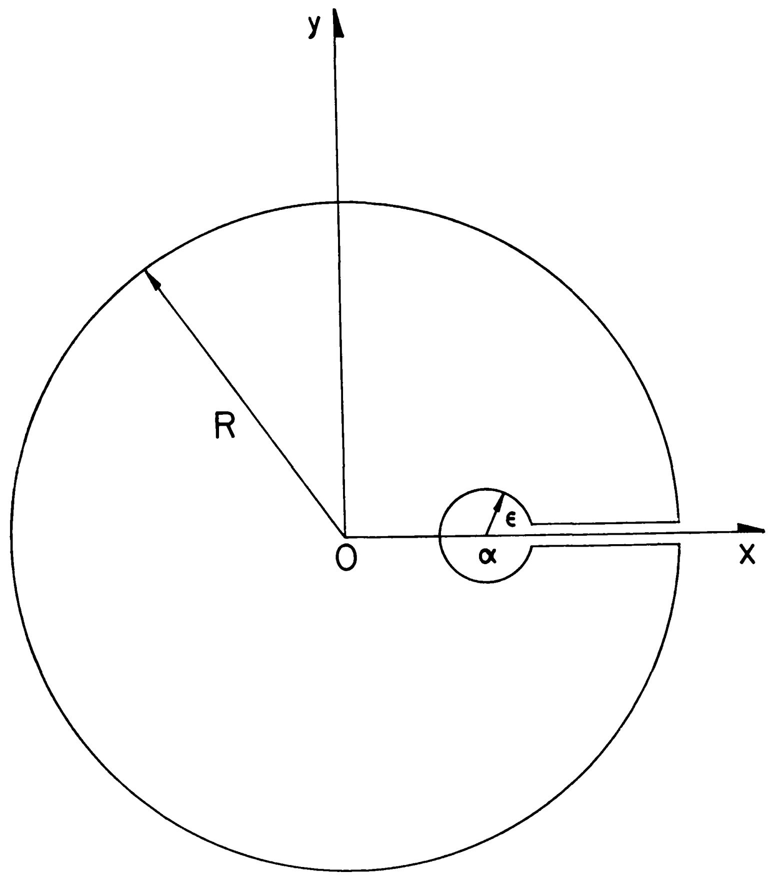

Fig. 2. The contour C.

$R \to \infty$ and $\varepsilon \to 0$. We find that if

$$M > \left(\int_\alpha^\infty + \int_\beta^\infty\right) \frac{1}{\sqrt{t-\alpha}} \frac{\mathrm{d}t}{t^2}, \tag{8}$$

then $\Omega(z)$ has no zeros, whereas if

$$M < \left(\int_\alpha^\infty + \int_\beta^\infty\right) \frac{1}{\sqrt{t-\alpha}} \frac{\mathrm{d}t}{t^2}, \tag{9}$$

then $\Omega(z)$ has precisely one real zero. This zero will be negative if $\Omega(0)<0$, i.e. if

$$M < \tfrac{2}{3}(\alpha^{-3/2} + \beta^{-3/2}), \tag{10}$$

which is the hyperbolic time criterion. Consequently if

$$\tfrac{2}{3}(\alpha^{-3/2} + \beta^{-3/2}) < M < \frac{\pi}{\alpha\sqrt{\alpha}} - \frac{1}{\beta\sqrt{\alpha}}\left(\sqrt{\frac{\beta}{\alpha} - 1} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right), \tag{11}$$

then the orbit is an ellipse.

If we now allow the possibility of the body having made k revolutions in addition to the arc P_1P_2, still with $S \in A$ and $S^* \notin A$ then clearly we require the zeros of

$$\Omega_k(z) = M - \frac{k\pi}{z\sqrt{z}} - \left(\int_\alpha^\infty + \int_\beta^\infty\right)\frac{1}{\sqrt{t - z}}\frac{dt}{t^2}. \tag{12}$$

On taking the principal branch of $\sqrt{z}$, we note that $\Omega_k(z)$ is analytic in the plane cut from $-\infty$ to 0 and from α to ∞, along the real axis. By employing the argument principle once more we deduce that $\Omega_k(z)$ has precisely one (real) zero if $\Omega_k(\alpha)>0$, i.e. if

$$M > \frac{k\pi}{\alpha\sqrt{\alpha}} + \left(\int_\alpha^\infty + \int_\beta^\infty\right)\frac{1}{\sqrt{t - \alpha}}\frac{dt}{t^2}, \qquad k = 1, 2, 3, \ldots. \tag{13}$$

If $\Omega_k(\alpha) \leqslant 0$, the function $\Omega_k(z)$ will have two zeros which will be complex if $|\Omega_k(\alpha)|$ is sufficiently large. We will consider this point in detail in the next section.

We now consider the case for elliptic orbits when $S \notin A$ but $S^* \in A$. Here we need to consider the function

$$\hat{\Omega}(z) = M - \frac{\pi}{z\sqrt{z}} + \left(\int_\alpha^\infty + \int_\beta^\infty\right)\frac{1}{\sqrt{t - z}}\frac{dt}{t^2}. \tag{14}$$

Again by the argument principle, we find that $\hat{\Omega}(z)$ has precisely one zero in the cut plane if $\hat{\Omega}(\alpha)>0$, i.e. if

$$M > \frac{\pi}{\alpha\sqrt{\alpha}} - \left(\int_\alpha^\infty + \int_\beta^\infty\right)\frac{1}{\sqrt{t - \alpha}}\frac{dt}{t^2}. \tag{15}$$

If we allow k complete revolutions in addition to the arc P_1P_2 but still with $S \notin A$ and $S^* \in A$, then the appropriate function is

$$\hat{\Omega}_k(z) = M - \frac{(k+1)\pi}{z\sqrt{z}} + \left(\int_\alpha^\infty + \int_\beta^\infty\right)\frac{1}{\sqrt{t - z}}\frac{dt}{t^2}, \tag{16}$$

which can be shown to have exactly one zero in the cut plane if $\hat{\Omega}_k(\alpha)>0$, i.e. if

$$M > \frac{(k+1)\pi}{\alpha\sqrt{\alpha}} - \left(\int_\alpha^\infty + \int_\beta^\infty\right)\frac{1}{\sqrt{t-\alpha}}\frac{\mathrm{d}t}{t^2}. \tag{17}$$

We shall now derive analytical expressions for the zeros of $\Omega(z)$, $\hat{\Omega}(z)$, $\Omega_k(z)$ and $\hat{\Omega}_k(z)$ for $k=1, 2, \ldots$. In so far as the analysis is concerned it is clear that we may combine the cases $\hat{\Omega}(z)$ and $\hat{\Omega}_k(z)$, $k=1, 2, 3, \ldots$, into one case, namely $\hat{\Omega}_k(z)$, $k=0, 1, 2, \ldots$.

3. Basic Analysis and Results

As was discussed in detail in I we now proceed to solve the transcendental equations of Section 2 by appealing to appropriately posed Riemann problems. We first consider Equation (6), assuming that inequality (9) holds. Clearly there exists an $F(z)$ which is analytic and non-zero in the cut plane such that

$$\Omega(z) = (z - z_0)\,F(z), \tag{18}$$

where z_0 is the required zero of $\Omega(z)$. If we take limiting values on the cut, we may write

$$F^+(x) = G(x)\,F^-(x), \qquad \alpha < x < \infty, \tag{19}$$

where

$$G(x) = \frac{\Omega^+(x)}{\Omega^-(x)}. \tag{20}$$

On using the terminology of Muskhelishvili (1953), we can say that $F(z)$ is a canonical solution of the Riemann problem defined by Equation (19). In fact we can write

$$F(z) = \frac{M}{z-\alpha}\exp\left[\frac{1}{\pi}\int_\alpha^\infty \arg\Omega^+(t)\frac{\mathrm{d}t}{t-z}\right], \tag{21}$$

and so from Equation (18) we have the identity

$$z_0 = z - \frac{\Omega(z)(z-\alpha)}{M}\exp\left[-\frac{1}{\pi}\int_\alpha^\infty \arg\Omega^+(t)\frac{\mathrm{d}t}{t-z}\right], \tag{22}$$

where the argument function has been chosen so that $\arg\Omega^+(\alpha) = -\pi$. Thus from Equation (3) we can now write

$$a = \tfrac{1}{2}M\left(Mz - \Omega(z)(z-\alpha)\exp\left[-\frac{1}{\pi}\int_\alpha^\infty \arg\Omega^+(t)\frac{\mathrm{d}t}{t-z}\right]\right)^{-1},$$

$$S \in A, \text{ or } S \in A \text{ and } S^* \notin A. \tag{23}$$

Although the choice of z in Equation (23) can alter the computational merit of that expression, an especially simple result is achieved by setting $z=0$:

$$a = \frac{M}{2\alpha}\left[M - \tfrac{2}{3}(\alpha^{-3/2} + \beta^{-3/2})\right]^{-1} \exp\left[\frac{1}{\pi}\int_{\alpha}^{\infty} \arg\Omega^{+}(t)\frac{\mathrm{d}t}{t}\right],$$
$$S\in A, \text{ or } S\in A \text{ and } S^{*}\notin A. \tag{24}$$

For computational purposes it is probably more expedient to make the change of variable $\tau=1/t$ so that we can write Equation (24) in the form

$$a = \frac{M}{2\alpha}\left[M - \tfrac{2}{3}(\alpha^{-3/2} + \beta^{-3/2})\right]^{-1} \exp\left[-\frac{1}{\pi}\int_{0}^{1/\beta} \tan^{-1}\left(\frac{2I_1(\tau) - I_3(\tau)}{M - \pi\tau^{3/2}}\right)\frac{\mathrm{d}\tau}{\tau} + \right.$$
$$\left. -\frac{1}{\pi}\int_{1/\beta}^{1/\alpha} \tan^{-1}\left\{\frac{I_1(\tau)}{M - \pi\tau^{3/2} + I_2(\tau)}\right\}\frac{\mathrm{d}\tau}{\tau}\right], \quad S\in A, \text{ or } S\in A \text{ and } S^{*}\notin A, \tag{25}$$

where the functions $I_1(\tau)$, $I_2(\tau)$ and $I_3(\tau)$ are as in I, namely

$$I_1(\tau) = \tau^{3/2}\left[\frac{\sqrt{1-\alpha\tau}}{\alpha\tau} + \tfrac{1}{2}\ln\left(\frac{1+\sqrt{1-\alpha\tau}}{1-\sqrt{1-\alpha\tau}}\right)\right], \tag{26a}$$

$$I_2(\tau) = \tau^{3/2}\left[\frac{\sqrt{\beta\tau-1}}{\beta\tau} + \tan^{-1}\sqrt{\beta\tau-1}\right], \tag{26b}$$

and

$$I_3(\tau) = I_1(\tau) - \tau^{3/2}\left[\frac{\sqrt{1-\beta\tau}}{\beta\tau} + \tfrac{1}{2}\ln\left(\frac{1+\sqrt{1-\beta\tau}}{1-\sqrt{1-\beta\tau}}\right)\right]. \tag{26c}$$

We now consider the case of elliptic motion, where we allow the body to make k complete revolutions in addition to the arc P_1P_2, but still with $S\in A$ and $S^{*}\notin A$, and, of course, when inequality (13) is satisfied. We now can write

$$z\Omega_k(z) = (z - z_k)\,F_k(z), \qquad k = 1, 2, 3, \ldots, \tag{27}$$

where $F_k(z)$ is the suitably normalized canonical solution of

$$F_k^{+}(x) = G_k(x)\,F_k^{-}(x), \qquad x\in(-\infty, 0)\cup(\alpha, \infty), \tag{28}$$

where

$$G_k(x) = \frac{\Omega_k^{+}(x)}{\Omega_k^{-}(x)}. \tag{29}$$

With $\arg\Omega_k^{+}(\pm\infty)=0$, we can write this solution as

$$F_k(z) = M\exp\left[\frac{1}{\pi}\left(\int_{-\infty}^{0} + \int_{\alpha}^{\infty}\right)\arg\Omega_k^{+}(t)\frac{\mathrm{d}t}{t-z}\right], \tag{30}$$

and so, from Equation (27) we have the identities

$$z_k = \left[z\,1 - \frac{\Omega_k(z)}{F_k(z)}\right], \qquad S \in A \text{ and } S^* \notin A, \tag{31}$$

and

$$a_k = \frac{1}{2z}\left[1 - \frac{\Omega_k(z)}{F_k(z)}\right]^{-1}, \qquad S \in A \text{ and } S^* \notin A. \tag{32}$$

On letting $z = \alpha$, we have the concise result

$$a_k = \frac{1}{2\alpha}\left[1 - \frac{\Omega_k(\alpha)}{F_k(\alpha)}\right]^{-1}, \qquad S \in A \text{ and } S^* \notin A, \tag{33}$$

where, after making appropriate variable changes,

$$\begin{aligned} F_k(\alpha) = M \exp\Bigg[&-\frac{1}{\pi}\int_{-1/\alpha}^{0} \tan^{-1}\left\{\frac{k\pi|\tau|^{3/2}}{(1-\alpha|\tau|)^{3/2}[M - I_5(\tau)]}\right\}\frac{d\tau}{\tau} + \\ &-\frac{1}{\pi}\int_{0}^{1/\beta} \tan^{-1}\left\{\frac{2I_1(\tau) - I_3(\tau)}{M - (k+1)\pi\tau^{3/2}}\right\}\frac{d\tau}{\tau(1-\alpha\tau)} + \\ &-\frac{1}{\pi}\int_{1/\beta}^{1/\alpha} \tan^{-1}\left\{\frac{I_1(\tau)}{M - (k+1)\pi\tau^{3/2} + I_2(\tau)}\right\}\frac{d\tau}{\tau(1-\alpha\tau)}\Bigg], \end{aligned} \tag{34}$$

and

$$\Omega_k(\alpha) = M - \alpha^{-3/2}\left[(k+1)\pi - \frac{\alpha}{\beta}\sqrt{\frac{\beta}{\alpha} - 1} - \tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right], \tag{35}$$

where $I_1(\tau)$, $I_2(\tau)$ and $I_3(\tau)$ are given by Equations (26), and

$$\begin{aligned} I_5(\tau) = &\frac{|\tau|^{1/2}}{\alpha(1-\alpha|\tau|)}\left[1 + \frac{\alpha}{\beta}\sqrt{1 + (\beta-\alpha)|\tau|}\right] + \\ &+ \frac{|\tau|^{3/2}}{2(1-\alpha|\tau|)^{3/2}}\left[\ln\left(\frac{1 - \sqrt{1-\alpha|\tau|}}{1 + \sqrt{1-\alpha|\tau|}}\right) + \right. \\ &\left. + \ln\left(\frac{\sqrt{1+(\beta-\alpha)|\tau|} - \sqrt{1-\alpha|\tau|}}{\sqrt{1+(\beta-\alpha)|\tau|} + \sqrt{1-\alpha|\tau|}}\right)\right]. \end{aligned} \tag{36}$$

An essentially similar analysis may be used to derive the desired solutions for $S \notin A$ and $S^* \in A$; thus we simply list

$$\hat{a}_k = \frac{1}{2\alpha}\left[1 - \frac{\hat{\Omega}_k(\alpha)}{\hat{F}_k(\alpha)}\right]^{-1}, \qquad S \notin A \text{ and } S^* \in A, \qquad k = 0, 1, 2\ldots, \tag{37}$$

where

$$\hat{\Omega}_k(\alpha) = M - \alpha^{-3/2}\left[k\pi + \frac{\alpha}{\beta}\sqrt{\frac{\beta}{\alpha} - 1} + \tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right], \quad k = 0, 1, 2, \ldots, \tag{38}$$

and

$$\hat{F}_k(\alpha) = M \exp\left[-\frac{1}{\pi}\int_{-1/\alpha}^{0} \tan^{-1}\left\{\frac{(k+1)\,\pi\,|\tau|^{3/2}}{(1-\alpha|\tau|)^{3/2}\,[M+I_5(\tau)]}\right\}\frac{d\tau}{\tau} + \right.$$
$$-\frac{1}{\pi}\int_{0}^{1/\beta} \tan^{-1}\left\{\frac{I_3(\tau)-2I_1(\tau)}{M-k\pi\tau^{3/2}}\right\}\frac{d\tau}{\tau(1-\alpha\tau)} +$$
$$\left. -\frac{1}{\pi}\int_{1/\beta}^{1/\alpha} \tan^{-1}\left\{\frac{-I_1(\tau)}{M-k\pi\tau^{3/2}-I_2(\tau)}\right\}\frac{d\tau}{\tau(1-\alpha\tau)}\right]. \quad (39)$$

We recall that inequality (15) has to be satisfied in order for the solution given by Equation (37) to be valid.

We now turn our attention to the multivalued case for $\Omega_k(z)$, $S\in A$ and $S^*\notin A$, $k\geqslant 1$. As was discussed previously if $\Omega_k(\alpha)<0$, it follows that $\Omega_k(z)$ has two zeros in the cut plane say z_{k0} and z_{k1}, so that we may write

$$z(z-\alpha)\,\Omega_k(z) = (z-z_{k0})\,(z-z_{k1})\,F_k(z), \quad (40)$$

where $F_k(z)$ is given by Equation (30). Thus on setting

$$K_k(z) = z(z-\alpha)\,\Omega_k(z)\,F_k^{-1}(z), \quad (41)$$

we have

$$z_{k0} = -B(v,\eta) - \sqrt{B^2(v,\eta) - C(v,\eta)}, \quad (42a)$$

and

$$z_{k1} = -B(v,\eta) + \sqrt{B^2(v,\eta) - C(v,\eta)}, \quad (42b)$$

where

$$B(v,\eta) = \tfrac{1}{2}\left[\frac{K_k(v)-K_k(\eta)-v^2+\eta^2}{v-\eta}\right], \quad (43a)$$

and

$$C(v,\eta) = \frac{vK_k(\eta)-\eta K_k(v)+v\eta(v-\eta)}{v-\eta}, \quad (43b)$$

with v and η being two distinct points off the cut. A convenient choice is for v and η to be real numbers selected from the interval $(0,\alpha)$. Consequently z_{k0} and z_{k1} will lead, through use of Equation (3), to appropriate semi-major axes a_{k0} and a_{k1} say, provided z_{k0} and z_{k1} are real. Clearly the limiting case will be when the two zeros coincide at x_k say. As this will be a double zero of $\Omega_k(z)$ it may be determined in terms of k, α and β by considering the zeros of $\Omega_k'(z)$. From Equation (12) we may write, after some elementary algebra,

$$\Omega_k'(z) = \frac{3k\pi}{2z^{5/2}} - \frac{1}{\alpha z\sqrt{\alpha-z}} - \frac{1}{\beta z\sqrt{\beta-z}} + \frac{3}{2z}\left(\int_{\alpha}^{\infty} + \int_{\beta}^{\infty}\right)\frac{1}{\sqrt{t-z}}\frac{dt}{t^2}. \quad (44)$$

If we now apply the argument principle to $\Omega_k'(z)$ in the same manner as it was applied in I to $\Lambda_k'(z)$ then we find that $\Omega_k'(z)$ can have either 3 or 5 zeros in the finite cut plane. However, in either case there is only one real zero in $(0, \alpha)$. The question as to which case it is, i.e. 3 or 5 zeros, can be established once values of k, α and β are specified. Consequently we quote only the relevant factorizations. For the case of 3 zeros we have

$$\Omega_k'(z) = \frac{(z - x_k)(z - z_{k1}')(z - \bar{z}_{k1}')}{z^3 (z - \alpha)\sqrt{\beta - z}} H_k(z), \tag{45}$$

and for the case of 5 zeros,

$$\Omega_k'(z) = \frac{(z - x_k)(z - z_{k1}')(z - \bar{z}_{k1}')(z - z_{k2}')(z - \bar{z}_{k2}')}{z^3 (z - \alpha)(z - \beta)^2 \sqrt{\beta - z}} H_k(z) \tag{46}$$

where z_{ki}' and $\bar{z}_{ki}'$, $i = 1, 2$, are the remaining zeros, not of interest here, and

$$H_k(z) = \tfrac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \exp\left\{\frac{1}{\pi}\int_{-\infty}^{0} \arg \Omega_k'^{+}(t) \frac{dt}{t - z} + \right.$$
$$\left. + \frac{1}{\pi}\int_{\alpha}^{\beta} \arg \Omega_k'^{+}(t) \frac{dt}{t - z} + \frac{1}{\pi}\int_{\beta}^{\infty} \left(\arg \Omega_k'^{+}(t) - \frac{\pi}{2}\right)\frac{dt}{t - z}\right\}. \tag{47}$$

Once the value of x_k has been determined, we may easily find the corresponding value of M, say M_k, from Equation (44) as

$$M_k = \tfrac{2}{3}\left\{\frac{1}{\alpha\sqrt{\alpha - x_k}} + \frac{1}{\beta\sqrt{\beta - x_k}}\right\}. \tag{48}$$

Consequently the condition for two real roots in this case will be, on taking inequality (13) into account,

$$\tfrac{2}{3}\left\{\frac{1}{\alpha\sqrt{\alpha - x_k}} + \frac{1}{\beta\sqrt{\beta - x_k}}\right\} < M < \frac{(k+1)\pi}{\alpha\sqrt{\alpha}} - \frac{1}{\beta\sqrt{\alpha}}\left\{\sqrt{\frac{\beta}{\alpha} - 1} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right\}. \tag{49}$$

Finally we collect together our principal results for the elliptic orbits. For the case in which the shaded region A contains neither the attracting focus nor the empty focus, depicted generically by Figure 3, the relevant inequality for M from I, is

$$\tfrac{2}{3}(\alpha^{-3/2} - \beta^{-3/2}) < M < \frac{1}{\beta\sqrt{\alpha}}\left[\sqrt{\frac{\beta}{\alpha} - 1} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right]. \tag{50}$$

For the case in which A contains S but not S^*, as shown in Figure 4, the inequality is

$$\tfrac{2}{3}(\alpha^{-3/2} + \beta^{-3/2}) < M < \frac{\pi}{\alpha\sqrt{\alpha}} - \frac{1}{\beta\sqrt{\alpha}}\left[\sqrt{\frac{\beta}{\alpha} - 1} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right]. \tag{51}$$

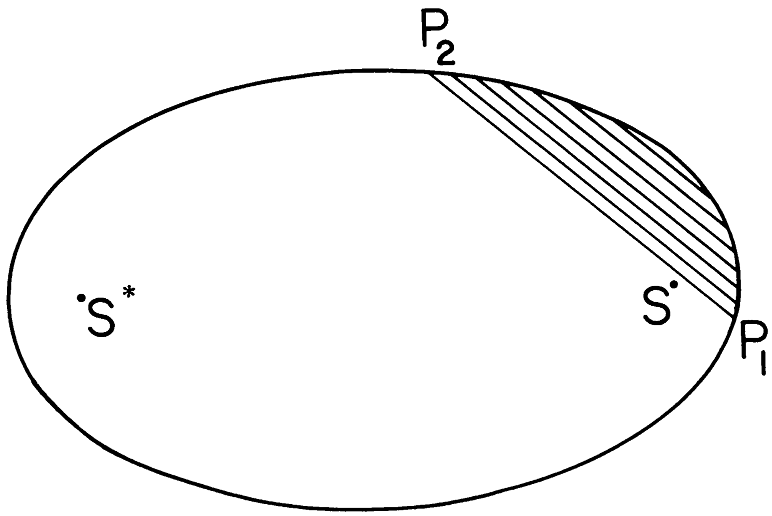

Fig. 3. Orbit basic to inequality (50).

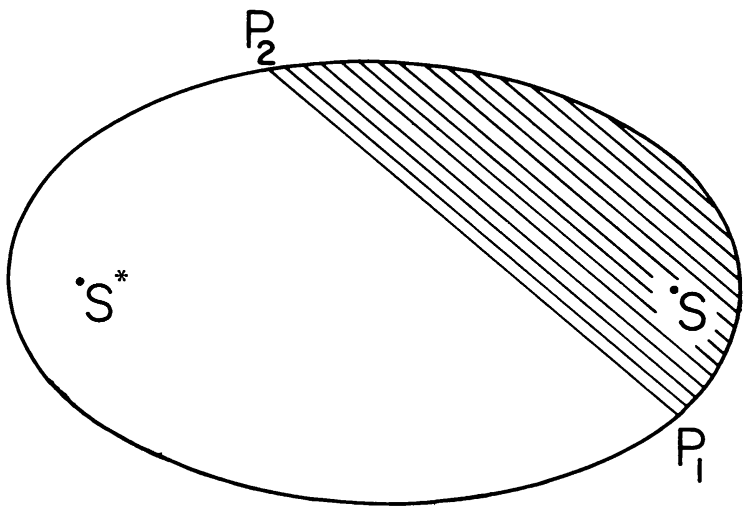

Fig. 4. Orbit basic to inequality (51).

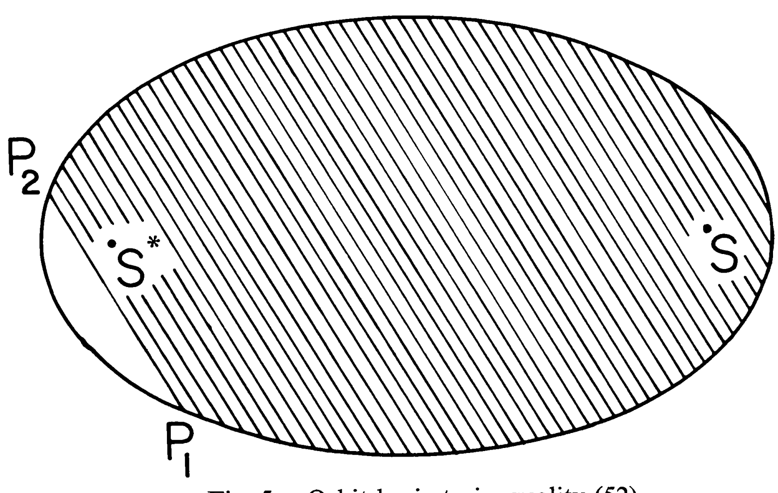

Fig. 5. Orbit basic to inequality (52).

If both S and S^* are included in A, as shown in Figure 5, the inequality for M from I is

$$\frac{\pi}{\alpha\sqrt{\alpha}} - \frac{1}{\beta\sqrt{\alpha}}\left[\sqrt{\frac{\beta}{\alpha} - 1} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right] < M. \tag{52}$$

Finally, for the case in which S^* is included in A, but S is not, see Figure 6, we have

$$\frac{1}{\beta\sqrt{\alpha}}\left[\sqrt{\frac{\beta}{\alpha} - 1} + \frac{\beta}{\alpha}\tan^{-1}\sqrt{\frac{\beta}{\alpha} - 1}\right] < M. \tag{53}$$

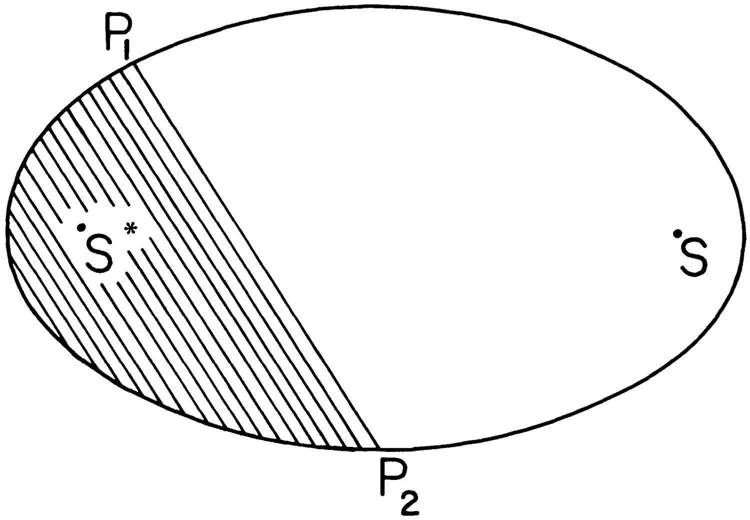

Fig. 6. Orbit basic to inequality (53).

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