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ON A PROBLEM OF UNIQUENESS REGARDING H-FUNCTION CALCULATIONS

C. E. SIEWERT*
Université des Sciences et Techniques, Lille, France

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Abstract—It is shown that the developed L equation, expecially useful for H-function calculations when ω is close to unity, has a unique solution.

1. INTRODUCTION

IT IS well known [see, for example, the work of MULLIKIN[1]] that the H function of CHANDRASEKHAR,[2]

$$H(\mu)=\frac{1+\mu}{(\nu_0+\mu)\sqrt{(1-\omega)}}\exp\left[-\frac{1}{\pi}\int_0^1\tan^{-1}\left(\frac{\omega\pi x}{2\{1-\omega x\tanh^{-1}x\}}\right)\frac{\mathrm{d}x}{x+\mu}\right],\quad \mu\in[0,1], \tag{1}$$

where ν_0 is the positive ($\omega<1$) zero of

$$\Lambda(z)=1+\frac{1}{2}\omega z\int_{-1}^{1}\frac{\mathrm{d}\mu}{\mu-z}, \tag{2}$$

is the unique solution of either

$$H(\mu)=1+\frac{\omega}{2}\mu H(\mu)\int_0^1 H(x)\frac{\mathrm{d}x}{x+\mu} \tag{3a}$$

and

$$\frac{\omega}{2}\nu_0\int_0^1 H(x)\frac{\mathrm{d}x}{\nu_0-x}=1, \tag{3b}$$

or

$$\lambda(\mu)H(\mu)=1+\frac{\omega}{2}\mu P\int_0^1 H(x)\frac{\mathrm{d}x}{x-\mu} \tag{4a}$$

and

$$\frac{\omega}{2}\nu_0\int_0^1 H(x)\frac{\mathrm{d}x}{\nu_0-x}=1, \tag{4b}$$

where the symbol P is used to denote integration in the Cauchy principal-value sense, and

$$\lambda(\mu)=1-\omega\mu\tanh^{-1}\mu. \tag{5}$$

It is clear that equation (1) can be evaluated to yield the desired numerical results for $H(\mu)$. In

*Permanent address: Nuclear Engineering Department, North Carolina State University, Raleigh, North Carolina, 27607 U.S.A.

many instances, however, researchers have elected to solve the non-linear integral equation (3a) by iteration in order to establish $H(\mu)$. Naturally, for such a scheme to be meaningful, the constraint, equation (3b), must be incorporated into the calculational scheme or, at least, verified in some way.

One of the basic problems associated with solving equation (3) iteratively is the rather slow convergence rate for ω close to unity. It is important that we be able to improve the convergence rate because for a more complicated problem, as, for example, that considered by BOND and SIEWERT,[3] an explicit solution for, say, the H matrix is not available, and thus there appears to be no alternative to an iterative method of computing $H(\mu)$. There are several other versions of equation (3a), and certainly some of them offer a greater convergence rate for ω close to unity. We wish to discuss here a method of computing $H(\mu)$ that has an immediate extension to the more difficult matrix problems.[3,4]

As noted by SHURE and NATELSON,[5] we can let

$$H(\mu)=\frac{\nu_0(1+\mu)}{\nu_0+\mu}L(\mu) \tag{6}$$

and subsequently rewrite equation (3a) as

$$L(\mu)=1+\mu L(\mu)\int_0^1 F(x)L(x)\frac{\mathrm{d}x}{x+\mu}, \tag{7}$$

after envoking the constraint given in equation (3b), here

$$F(x)=\frac{\omega}{2}\frac{\nu_0^2(1-x^2)}{\nu_0^2-x^2}. \tag{8}$$

We have found from experience that, for $\omega \in [0{\cdot}5, 1{\cdot}0]$, an iterative solution of equation (7) converges rapidly and can be used in equation (6) to yield the correct $H(\mu)$. Though this procedure has proved to be an efficient way of computing the H function, two questions should now be answered: (a) is the H function computed in this way a solution of equations (3a) and (3b)? and (b) is the solution of equation (7) unique?

2. ANALYSIS

We now wish to show that the equation

$$L(\mu)=1+\mu L(\mu)\int_0^1 F(x)L(x)\frac{\mathrm{d}x}{x+\mu},\quad \mu\in[0,1], \tag{9}$$

has a unique solution in the class of continuous functions for $\mu \in [0, 1]$. If we multiply equation (9) by

$$1+\mu P\int_0^1 F(\nu)L(\nu)\frac{\mathrm{d}\nu}{\nu-\mu}$$

and carry out some elementary algebraic operations, we find that the resulting equation can be written as

$$\frac{2}{\omega}F(\mu)\lambda(\mu)L(\mu)=1+\mu P\int_0^1 F(x)L(x)\frac{\mathrm{d}x}{x-\mu}. \tag{10}$$

Equation (10) is clearly a singular integral equation for $L(\mu)$. If we let

$$h(\mu)=\frac{2}{\omega}F(\mu)L(\mu) \tag{11}$$

then we can write equation (10) in the familiar form

$$-\frac{\omega}{2}\mu P\int_0^1 h(x)\frac{\mathrm{d}x}{x-\mu}+\lambda(\mu)h(\mu)=1. \tag{12}$$

The analysis of MUSKHELISHVILI[6] can now be used to solve equation (12) to obtain the general result

$$h(\mu)=\frac{(\nu_0-\beta\mu)}{\nu_o-\mu}H(\mu), \tag{13}$$

where $H(\mu)$ is given by equation (1) and β is an arbitrary constant. Equation (13) represents a one-parameter family of solutions to equation (12). If we now enter equation (13) into equation (11), we find

$$L(\mu)=\frac{(\nu_0-\beta\mu)(\nu_0+\mu)}{\nu_0^2(1-\mu^2)}H(\mu). \tag{14}$$

Clearly then, if $L(\mu)$ is to be continuous for $\mu\in[0,1]$, we must take $\beta=\nu_0$, and thus we find the only continuous solution to equation (9) to be

$$L(\mu)=\frac{(\nu_0+\mu)H(\mu)}{\nu_o(1+\mu)}. \tag{15}$$

Also, since equation (15) is identical to equation (6), we conclude that the solution of the L equation will yield, by way of equation (6), the correct $H(\mu)$.

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REFERENCES

1. T. W. MULLIKIN, *Trans. Am. Math. Soc.* **113**, 316 (1964).
2. S. CHANDRASEKHAR, *Radiative Transfer.* Oxford University Press, London (1950).
3. G. R. BOND and C. E. SIEWERT, *Astrophys. J.* **164**, 97 (1971).
4. J. T. KRIESE and C. E. SIEWERT, *Astrophys. J.* **164**, 389 (1971).
5. F. SHURE and M. NATELSON, *Anls. Phys.* **26**, 274 (1964).
6. N. I. MUSKHELISHVILI, *Singular Integral Equations.* Noordhoff, Groningen, The Netherlands (1953).