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ON PARTICULAR SOLUTIONS OF THE EQUATION OF TRANSFER

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Abstract—The particular solutions corresponding to two very special inhomogeneous terms of the equation of transfer with linearly anisotropic scattering are reported

I INTRODUCTION

IT IS clear that the analysis of radiative transfer problems, based on the inhomogeneous equation

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} (1 + b\mu\mu')I(\tau, \mu')\, d\mu' + S(\tau), \tag{1}$$

can proceed along classical lines[1] once a particular solution $I_p(\tau, \mu)$, corresponding to a given inhomogeneous source term $S(\tau)$, is established. In this paper, we wish to report two particular solutions which correspond to source terms explicitly omitted in an existing tabulation[2] of particular solutions, namely

$$S(\tau) = e^{-\tau/\eta_0} \quad \text{and} \quad S(\tau) = \tau\, e^{-\tau/\eta_0} \tag{2}$$

where η_0 denotes a "discrete eigenvalue" of the homogeneous equation of transfer:

$$\Lambda(\eta_0) = \omega R(\eta_0, \eta_0) \left[1 - \eta_0 \tanh^{-1} \frac{1}{\eta_0}\right] + 1 - \omega = 0, \tag{3}$$

where

$$R(x, y) = 1 + b(1 - \omega)xy \tag{4}$$

II ANALYSIS

If we first consider $S(\tau) = \exp(-\tau/\eta_0)$ and substitute

$$I_p(\tau, \mu) = \varphi(\eta_0, \mu)\, e^{-\tau/\eta_0} [A\tau + B(\mu)], \tag{5}$$

where

$$\varphi(\eta_0, \mu) = \frac{\omega}{2} \eta_0 \frac{R(\eta_0, \mu)}{\eta_0 - \mu}, \tag{6}$$

into eqn (1), we find that the constant A and the unknown $B(\mu)$ should satisfy

$$A\mu\varphi(\eta_0, \mu) + \frac{1}{2} \omega R(\eta_0, \mu) B(\mu) = 1 + \frac{\omega}{2} \int_{-1}^{1} B(\mu')\varphi(\eta_0, \mu')(1 + b\mu\mu')\, d\mu'. \tag{7}$$

Multiplying eqn (7) by $\varphi(\eta_0, \mu)$ and integrating over μ from -1 to 1, we find immediately that

$$\frac{1}{A} = N(\eta_0) = \int_{-1}^{1} \mu\varphi^2(\eta_0, \mu)\, d\mu. \tag{8}$$

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To find $B(\mu)$, we can first integrate eqn (7) from -1 to 1 to obtain

$$A\eta_0(1-\omega)+\frac{1}{2}\omega\int_{-1}^{1} R(\eta_0,\mu)B(\mu)\,d\mu = 2+\omega B_0, \tag{9}$$

where, in general,

$$B_\alpha = \int_{-1}^{1} \varphi(\eta_0,\mu)B(\mu)\mu^\alpha\,d\mu. \tag{10}$$

Thus, after noting that

$$B_1 = \eta_0 B_0 - \frac{1}{2}\omega\eta_0\int_{-1}^{1} R(\eta_0,\mu)B(\mu)\,d\mu, \tag{11}$$

we can write eqn (7) as

$$A\mu\varphi(\eta_0,\mu)+\frac{1}{2}\omega R(\eta_0,\mu)B(\mu) = 1+\frac{1}{2}\omega R(\eta_0,\mu)B_0+\frac{1}{2}\omega b\eta_0\mu[-2+A\eta_0(1-\omega)]. \tag{12}$$

Since $\varphi(\eta_0,\mu)\exp(-\tau/\eta_0)$ is a solution of the homogeneous version of eqn (1), it is clear that the constant B_0 appearing in eqn (12) can be chosen arbitrarily. We now take $B_0 = -2/\omega$ to obtain the final result

$$B(\mu) = \frac{-2}{\omega R(\eta_0,\mu)}\left\{A\mu\varphi(\eta_0,\mu)+b\eta_0\mu\left[1-\frac{1}{2}\omega A\eta_0(1-\omega)\right]\right\}. \tag{13}$$

The algebra associated with the case $S(\tau)=\tau\exp(-\tau/\eta_0)$ is considerably more tedious, and thus we will only list here the final results:

$$I_p(\tau,\mu) = \varphi(\eta_0,\mu)e^{-\tau/\eta_0}[C\tau^2+D(\mu)\tau+E(\mu)], \tag{14}$$

where

$$C = \frac{1}{2N(\eta_0)}, \tag{15}$$

$$D(\mu) = -\frac{4C}{\omega R(\eta_0,\mu)}\left[\mu\varphi(\eta_0,\mu)-\omega R(\eta_0,\mu)\left(\frac{L(\eta_0)}{N(\eta_0)}-K(\eta_0)\right)-N(\eta_0)\times\{1-\omega b\eta_0\mu[1-C\eta_0(1-\omega)]\}\right], \tag{16}$$

and

$$E(\mu) = \frac{2}{\omega R(\eta_0,\mu)}\left[-\mu\varphi(\eta_0,\mu)D(\mu)+\frac{\omega}{2}b\eta_0\mu R\right]. \tag{17}$$

Here we have let

$$L(\eta_0) = \int_{-1}^{1}\frac{\varphi^3(\eta_0,\mu)}{\omega R(\eta_0,\mu)}\mu^2\,d\mu, \tag{18}$$

$$K(\eta_0) = \int_{-1}^{1}\frac{\varphi^2(\eta_0,\mu)}{\omega R(\eta_0,\mu)}\{1+\omega b\eta_0\mu[-1+C\eta_0(1-\omega)]\}\mu\,d\mu, \tag{19}$$

and

$$R = \frac{\eta_0(1-\omega)}{N(\eta_0)}\left[\eta_0-\left(\frac{2}{1-\omega}\right)N(\eta_0)+2\frac{L(\eta_0)}{N(\eta_0)}-2K(\eta_0)\right]. \tag{20}$$

We note that Ref. (2) contains particular solutions for inhomogeneous source terms of the form $\exp(-\tau/\eta)$ and $\tau \exp(-\tau/\eta)$, $\eta \neq \eta_0$, and thus the results given here complete the tabulation of Ref. (2).

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