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# Technical Notes

## On the Critical Reactor Problem for a Reflected Slab

C. E. Siewert and A. R. Burkart

*North Carolina State University*
*Department of Nuclear Engineering*
*Raleigh, North Carolina 27607*

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ABSTRACT

*The elementary solutions of the one-speed transport equation are used, along with certain invariance principles, to establish tractable solutions to reflected reactor critical problems in plane geometry.*

## I. INTRODUCTION

A paper by Kuszell[1] was the first to use the elementary solutions and associated analysis, newly developed by Case,[2] to study critical problems, in one-group theory, for reflected slabs. In the paper by Kuszell, the critical problem for an infinitely reflected slab was reduced to a set of integral equations, later solved numerically by Kowalska.[3] In the present Note, we wish to show how the invariance principles discussed by Chandrasekhar[4] can be used to simplify greatly the analysis of the two-media critical problem in plane geometry.

We consider the one-speed transport equations for the core, $-a \le x \le a$, and the reflector $|x| > a$, written in the familiar[5] manner

$$\mu \frac{\partial}{\partial x} \Psi_\alpha(x,\mu) + \Psi_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} \Psi_\alpha(x,\mu')\,d\mu' \ , \qquad \alpha = 1 \text{ and } 2 \ , \tag{1}$$

where $\alpha = 1$ implies the core and $\alpha = 2$ implies the reflector. Clearly we take $c_1 > 1$ and $c_2 < 1$. We thus seek solutions of Eq. (1) such that $\Psi_\alpha(-x,-\mu) = \Psi_\alpha(x,\mu)$, $\Psi_1(a,\mu) = \Psi_2(a,\mu)$, and $\Psi_2(\infty,\mu) = 0$. We consider that $c_1$ and $c_2$ are given and thus seek, in addition to $\Psi_\alpha(x,\mu)$, $\alpha = 1$ and 2, the critical half-thickness, $a$.

## II. ANALYSIS

For the core region, we can write the solution to Eq. (1) in a standard[5] notation as

$$\Psi_1(x,\mu) = A(\nu_0)\,[\Phi_1(\nu_0,\mu)\exp(-x/\nu_0) + \Phi_1(-\nu_0,\mu)\exp(x/\nu_0)] + \int_0^1 A(\nu)\,[\Phi_1(\nu,\mu)\exp(-x/\nu) + \Phi_1(-\nu,\mu)\exp(x/\nu)]d\nu \ . \tag{2}$$

The solution given by Eq. (2) in terms of Case's elementary solutions clearly satisfies the necessary symmetry condition. We need only consider $x > a$ for the reflector region, and thus, in a similar manner, we can write the solution in the reflector as

$$\Psi_2(x,\mu) = B(\eta_0)\Phi_2(\eta_0,\mu)\exp(-x/\eta_0) + \int_0^1 B(\eta)\Phi_2(\eta,\mu)\exp(-x/\eta)\,d\eta \ , \qquad x > a \ . \tag{3}$$

Note that the arbitrary expansion coefficients $A(\nu_0)$, $A(\nu)$, $B(\eta_0)$, and $B(\eta)$, as well as the critical half-thickness, $a$, are now to be determined by applying the continuity condition at $x = a$. In the past,[1,6,7] Eqs. (2) and (3) have been evaluated at $x = a$ and equated; the resulting singular integral equation was then[1,6,7] regularized and solved numerically.

We would like now to proceed somewhat differently and subsequently make use of an idea of Ishiguro[8] to introduce the $S$ function developed by Chandrasekhar[4,9] from basic invariance principles. We can first write

$$\Psi_1(a,\mu) = \Psi_2(a,\mu) \ , \qquad \mu > 0 \ ,$$

and

$$\Psi_1(a,-\mu) = \Psi_2(a,-\mu) \ , \qquad \mu > 0 \ .$$

However, we can use Chandrasekhar's $S$ function[4]

$$S_2(\mu',\mu) = c_2 \frac{\mu\mu'}{\mu'+\mu} H_2(\mu')H_2(\mu) \tag{4}$$

to write

$$\Psi_2(a,-\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\Psi_2(a,\mu')\,d\mu' \ ; \tag{5}$$

and thus, because of the continuity conditions,

$$\Psi_1(a,-\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\Psi_2(a,\mu')\,d\mu' \ , \qquad \mu > 0 \ , \tag{6}$$

or

$$\Psi_1(a,-\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\Psi_1(a,\mu')\,d\mu' \ , \qquad \mu > 0 \ . \tag{7}$$

In Eq. (4), $H_2(\mu)$ is the Chandrasekhar[4] $H$ function for the reflector; since we will also need the $H$ function for the core, we give

$$H_\alpha(\mu) = \frac{1+\mu}{(x_\alpha+\mu)(1-c_\alpha)^{1/2}} \exp\left\{-\frac{1}{\pi} \times \int_0^1 \tan^{-1}\left[\frac{\pi x c_\alpha}{2(1-c_\alpha x \tanh^{-1}x)}\right]\frac{dx}{x+\mu}\right\} \ , \tag{8}$$

where $x_1 = \nu_0$ and $x_2 = \eta_0$.

We can now substitute Eq. (2) into Eq. (7) to obtain the coefficients $A(\nu_0)$ and $A(\nu)$ and the half-thickness, $a$. On making this substitution and evaluating several ensuing integrals, we find we can write the resulting equation in the tractable form

$$\frac{A(\nu_0)}{H_2(\nu_0)}\exp(a/\nu_0)\Phi_1(\nu_0,\mu) + \int_0^1 \frac{A(\nu)}{H_2(\nu)}\exp(a/\nu)\Phi_1(\nu,\mu)\,d\nu = -A(\nu_0)\left(1-\frac{c_2}{c_1}\right)H_2(\nu_0)\exp(-a/\nu_0)\Phi_1(-\nu_0,\mu) - \int_0^1 A(\nu)\left(1-\frac{c_2}{c_1}\right)H_2(\nu)\exp(-a/\nu)\Phi_1(-\nu,\mu)\,d\nu \ , \qquad \mu \in (0,1) \ . \tag{9}$$

Equation (9) is a singular integral equation that we can readily regularize using the orthogonality properties of the elementary solutions.[5] Before proceeding, however, we wish to point out that the only differences between our Eq. (9) and the equation obtained by Mitsis[10] for the bare-slab problem is the presence of $H_2(\xi)$ and the factor $[1-(c_2/c_1)]$ in Eq. (9). In fact, for the vacuum limit, $H_2(\xi)$ goes to 1 and $c_2$ goes to 0, and thus Eq. (9) reduces immediately to the Mitsis equation.[10]

If we now multiply Eq. (9) by $\mu H_1(\mu)\Phi_1(\nu_0,\mu)$, integrate over $\mu$, and use the orthogonality relations of Kuščer et al.,[11] we find

$$E(\nu_0)\left[\frac{N_1(\nu_0)}{H(\nu_0)} + H(\nu_0)\frac{(c_1-c_2)\nu_0}{4}\exp(-2a/\nu_0)\right] = -\int_0^1 E(\nu')H(\nu')\frac{(c_1-c_2)\nu'\nu_0}{2(\nu'+\nu_0)}\exp(-2a/\nu')\,d\nu' \ , \tag{10}$$

[1]A. KUSZELL, *Acta Physica Polon.*, **20**, 567 (1961).
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[7]N. J. McCORMICK and R. J. DOYAS, *Nucl. Sci. Eng.*, **37**, 308 (1969).
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[10]G. J. MITSIS, *Nucl. Sci. Eng.*, **17**, 55 (1963).
[11]I. KUŠČER, N. J. McCORMICK, and G. C. SUMMERFIELD, *Ann. Phys.*, **30**, 411 (1964).

where

$$N_1(\nu_0) = \frac{c_1}{2}\nu_0^3\left(\frac{c_1}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right) , \quad (11)$$

$$H(\xi) = \frac{H_2(\xi)}{H_1(\xi)} , \quad \xi = \nu_0 \text{ or } \xi \in (0,1) , \quad (12)$$

and

$$E(\xi) = A(\xi)\exp(a/\xi) , \quad \xi = \nu_0 \text{ or } \xi \in (0,1) . \quad (13)$$

In a similar manner, we can multiply Eq. (9) by $\mu H_1(\mu)\Phi_1(\nu,\mu)$ and integrate over $\mu$ to obtain the regular integral equation

$$E(\nu)\frac{N_1(\nu)}{H(\nu)} = -E(\nu_0)H(\nu_0)\frac{(c_1 - c_2)\nu\nu_0}{2(\nu + \nu_0)}\exp(-2a/\nu_0) - \int_0^1 E(\nu')H(\nu')\frac{(c_1 - c_2)\nu\nu'}{2(\nu + \nu')}\exp(-2a/\nu')d\nu' , \quad \nu \in (0,1) , \quad (14)$$

where

$$N_1(\nu) = \nu\left[(1 - c_1\nu\tanh^{-1}\nu)^2 + \left(\frac{c_1\nu\pi}{2}\right)^2\right] . \quad (15)$$

## III. CONCLUSION

The problem of finding the critical half-thickness of an infinitely reflected slab reactor has been reduced to the regular integral equations,

$$\exp(-2a/\nu_0) = -\frac{4}{H(\nu_0)(c_1 - c_2)\nu_0} \times\left[\frac{N_1(\nu_0)}{H(\nu_0)} + \int_0^1 E(\nu')H(\nu')\frac{(c_1 - c_2)\nu'\nu_0}{2(\nu' + \nu_0)}\exp(-2a/\nu')d\nu'\right] \quad (16a)$$

and

$$E(\nu) = -\frac{H(\nu)}{N_1(\nu)}\frac{(c_1 - c_2)\nu}{2}\left[H(\nu_0)\frac{\nu_0}{(\nu + \nu_0)}\exp(-2a/\nu_0) + \int_0^1 E(\nu')H(\nu')\frac{\nu'}{(\nu + \nu')}\exp(-2a/\nu')d\nu'\right] , \quad \nu \in (0,1) , \quad (16b)$$

developed in Sec. II [note that we have arbitrarily imposed the normalization $E(\nu_0) = 1$]. Equations (16) are similar but not apparently identical to the results of Leuthäuser,[12] who used the standard technique of Muskhelishvili[13] to regularize his singular equations. Note that in Eqs. (16), as well as in Eq. (9), the reduction to the vacuum limit is accomplished by setting $c_2 = 0$ and $H_2(\xi) = 1$. It is thus apparent that an existing program written to solve the bare-slab case could very easily be modified to solve this reflected case. This we have done, and we found that for the cases considered, an iterative solution of Eqs. (16) converged rapidly to yield results we believe correct to six significant figures, though agreement with the Kowalska[3] results was, in general, established only to two or three significant figures.

It is clear that an "asymptotic" solution to the considered problem can be obtained immediately by ignoring Eq. (16b) and taking $E(\nu) = 0$ in Eq. (16a). We find a critical condition identical to the bare case, except that the extrapolated endpoint is defined differently:

[12]K.-D. LEUTHÄUSER, *Atomkernenergie*, **14**, 171 (1969).

[13]N. I. MUSKHELISHVILI, *Singular Integral Equations*, Nordhoff, Groningen (1953).

$$a_0 = \tfrac{1}{2}\pi|\nu_0| - z_0 , \quad (17)$$

where

$$z_0 = \frac{\nu_0}{2}\operatorname{Log}\left[\frac{4N_1(\nu_0)}{H^2(\nu_0)(c_1 - c_2)\nu_0}\right] . \quad (18)$$

Equation (18) is equivalent to the expression reported by Leuthäuser[14] for the Milne extrapolated endpoint for adjacent half-spaces. Note that Log is used in Eq. (18) to denote the principal branch of the log function. Numerical results obtained for $a_0$ and $z_0$ are in agreement with the results of Leuthäuser[15] and of Doyas and McCormick[16]; our exact results agree exactly with Caroll and Aronson.[17] In addition to providing an excellent general discussion of this problem, the review of McCormick and Kuščer[18] contains a summary of other methods of solution and approximation techniques.

Though our final results differ only slightly from the Leuthäuser[12] equations, we believe the method used to establish them is more concisely expressed and will prove of value for the two- or multigroup version of this problem or one-group problems with anisotropic scattering.

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