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One-Speed Transport Theory for Spherical Media with Internal Sources and Incident Radiation

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1. Introduction

A great deal of interest in neutron-transport theory is centered about the use of Case's singular eigenfunction expansion technique [1, 2] for problems with plane symmetry, and the successful applications have been numerous; however, because of inherent difficulties, even the one-speed problems in neutron-transport theory for non-planar geometries have not been solved as readily, nor as rigorously, as might be desired. Some progress, however, has been made. Davison [3] and Mitsis [4] established both the singular and the regular solutions of the homogeneous transport equation in spherical and cylindrical geometries, and the latter author solved thoroughly the critical problem for the bare sphere and the unreflected infinite cylinder.

Leonard and Mullikin [5] have discussed the plane- and spherical-geometry resolvent kernel for the Fredholm equation, the solution of which is the neutron density. The transform technique and the eigenfunctions of the homogeneous transport equation were used by Erdmann and Siewert [6] to develop Green's functions of interest for spherical problems in finite and infinite media. Shani [7] and Smith [8] have discussed the so-called black-sphere problem for one-speed theory, and Gibbs [9] has utilized a general method for convex media; the latter technique, however, has been applied principally to problems without an incident neutron flux.

The purpose of the present work is to extend the transform method used by Mitsis [4] to include the effects of sources and, more importantly, neutrons incident on the free surface of a finite sphere. The analysis is carried out for the time-independent one-speed model, and the sources and the surface boundary conditions are, in general, arbitrary. The mathematical model discussed herein is applicable to the development of many reactor concepts including high temperature reactors with spherical fuel elements [10], AVR reactors [11], liquid fluidized beds of spheres [12], and the so-called pebble bed reactors [13, 14].

2. Basic Analysis

Assuming an isotropic scattering law, we seek a solution to the steady-state one-speed neutron transport equation, written in the spherically symmetric form

$$\mu \frac{\partial}{\partial r} \Psi(r, \mu) + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} \Psi(r, \mu) + \Psi(r, \mu) = \frac{c}{2} \int_{-1}^{+1} \Psi(r, \mu') d\mu' + S(r), \tag{2.1}$$

subject to the free-surface boundary condition

$$\Psi(R, -\mu) = F(\mu), \qquad \mu\varepsilon(0, 1). \tag{2.2}$$

Here $\Psi(r, \mu)$ denotes the neutron angular density in a sphere of radius R (in optical units), r is the optical variable, μ is the direction cosine of the propagating neutron, $c < 1$ is the mean number of secondary neutrons per collision, and $S(r)$ represents the inhomogeneous source term. Further, the function $F(\mu)$, $\mu\varepsilon(0, 1)$, describes the arbitrary incident distribution of neutrons.

From equation (2.1) we can deduce the integral equation for the neutron density [3]:

$$\begin{aligned}\rho(r) = &\int_{-1}^{+1} F(|\mu_0(r, \mu')|)e^{-s_0(r,\mu')}d\mu' \\ &+ \frac{1}{r}\int_{-R}^{+R} r'E_1(|r - r'|)\left[S(r') + \frac{c}{2}\rho(r')\right]dr',\end{aligned} \tag{2.3}$$

where we have extended the range of r to $r\varepsilon[-R, R]$, and have defined $\rho(-r) = \rho(r)$ and $S(-r) = S(r)$. Here the first exponential integral function is denoted by $E_1(x)$.

If we now define two transform functions,

$$\Phi(r, \mu) = \frac{1}{\mu}\int_{-R}^{r} dr'r'e^{-(r-r')/\mu}\left[S(r') + \frac{c}{2}\rho(r')\right], \qquad \mu\varepsilon(0, 1), \tag{2.4a}$$

and

$$\Phi(r, -\mu) = \frac{1}{\mu}\int_{r}^{R} dr'r'e^{-(r'-r)/\mu}\left[S(r') + \frac{c}{2}\rho(r')\right], \qquad \mu\varepsilon(0, 1), \tag{2.4b}$$

we observe the equation (2.3) can be written as

$$\begin{aligned}\rho(r) = &\int_{-1}^{+1} F(|\mu_0(r, \mu')|)e^{-s_0(r,\mu')}d\mu' \\ &+ \frac{1}{r}\int_{0}^{1} [\Phi(r, \mu') + \Phi(r, -\mu')]d\mu', \qquad r\varepsilon[-R, R].\end{aligned} \tag{2.5}$$

We note that $\Phi(r, \mu)$, $\mu\varepsilon(-1, 1)$, has been defined differently for positive and negative μ; however, differentiation of equations (2.4) and (2.5) may be used to show that in general $\Phi(r, \mu)$ must be a solution of

$$\begin{aligned}\mu\frac{\partial}{\partial r}\Phi(r, \mu) + \Phi(r, \mu) = &\frac{c}{2}\int_{-1}^{+1} \Phi(r, \mu')d\mu' + rS(r) \\ &+ \frac{cr}{2}\int_{-1}^{+1} F(|\mu_0(r, \mu')|)e^{-s_0(r,\mu')}d\mu'.\end{aligned} \tag{2.6}$$

Equation (2.6) is, of course, the differential transport equation appropriate for applications in plane geometry, with an inhomogeneous forcing function

$$Q(r) = rS(r) + \frac{cr}{2}\int_{-1}^{+1} F(|\mu_0(r, \mu')|)e^{-s_0(r,\mu')}d\mu'. \tag{2.7}$$

The boundary conditions subject to which a solution to equation (2.6) must be constructed follow immediately from the definitions given by equations (2.4):

$$\Phi(r, \mu) = -\Phi(-r, -\mu) \tag{2.8a}$$

and

$$\Phi(R, -\mu) = 0, \qquad \mu\varepsilon(0, 1). \tag{2.8b}$$

It is evident that the original problem of determining $\rho(r)$ has been reduced to the need to solve the pseudo slab problem defined by equations (2.6) and (2.8); clearly once this is accomplished, $\rho(r)$ follows simply from equation (2.5).

3. Particular Solutions

It is clear that the pseudo problem defined by equations (2.6) and (2.8) can be solved by Case's method once a particular solution appropriate to the inhomogeneous source term $Q(r)$ is established. Clearly the required particular solution can always be written in terms of the infinite-medium Green's function; however, in order to establish more tractable particular solutions for explicit inhomogeneous terms, several particular cases can be mentioned. The simplest case, of course, is $F(\mu) = 0$, in which case only the internal source for the sphere need to be considered.

The case of a constant source in a finite sphere without incident neutrons leads to

$$Q(r) = r, \tag{3.1}$$

for which the appropriate particular solution is [15]

$$\Phi_p(r, \mu) = \frac{1}{1-c}(r-\mu), \qquad c \neq 1, \tag{3.2a}$$

$$= \mu^3 - \frac{1}{2}(r-\mu)[(r-\mu)^2 + 3\mu^2], \qquad c = 1. \tag{3.2b}$$

Particular solutions corresponding to isotropically emitting sources described by arbitrary polynomials of r have been derived by Lundquist and Horak [15], so that further examples of this type are readily available.

In regard to incident neutrons, the case of a constant $F(\mu)$, say unity, is the simplest, but perhaps the most interesting for reactor physics calculations. Here we need to evaluate

$$Q(r) = \frac{cr}{2}\int_{-1}^{+1} e^{-s_0(r,\mu')}d\mu', \tag{3.3}$$

which integrates to

$$Q(r) = \frac{c}{2}\{R[E_2(R-r) - E_2(R+r)] + [E_3(R-r) - E_3(R+r)]\}, \tag{3.4}$$

where $E_2(x)$ and $E_3(x)$ are higher order exponential integral functions. Özişik and Siewert [16] have found that for an inhomogeneous source term of the form

$$Q(r) = E_N(R+r), \tag{3.5}$$

Equation (2.6) accepts the solution

$$\Phi_p^{(N)}(r, \mu) = -\frac{2}{c}\mu^{N-2}e^{-(R+r)/\mu}, \qquad \mu\varepsilon(0, 1), \tag{3.6a}$$

$$= 0, \qquad \mu\varepsilon(-1, 0). \tag{3.6b}$$

Similarly, for $Q(r) = E_N(R-r)$, we note

$$\Phi_p^{(N)}(r, \mu) = 0, \qquad \mu\varepsilon(0, 1), \tag{3.7a}$$

$$= -\frac{2}{c}(-\mu)^{N-2}e^{(R-r)/\mu}, \qquad \mu\varepsilon(-1, 0). \tag{3.7b}$$

These results can thus be used to deduce the particular solution corresponding to $Q(r)$ as given by equation (3.4). Hence,

$$\Phi_p(r, \mu) = (R + \mu)e^{-(R+r)/\mu}, \qquad \mu\varepsilon(0, 1), \tag{3.8a}$$

$$= -(R - \mu)e^{(R-r)/\mu}, \qquad \mu\varepsilon(-1, 0). \tag{3.8b}$$

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Abstract

The time-independent one-speed neutron transport equation for media with spherically symmetric invariance properties and an isotropic scattering law is discussed. The analysis allows for isotropic internal emission, and emphasis is placed on the treatment of neutrons incident on the free surface of a finite spherical medium.

Zusammenfassung

Gegenstand diese Artikels ist eine zeitunabhängige, monoenergetische Neutronentransportgleichung für Medien mit sphärisch symmetrischen Invarianzeigenschaften und isotropem Streugesetz. Die Analysis erlaubt isotrope interne Emission und behandelt das Problem des Neutroneneinfalls auf eine freie Oberfläche eines endlichen sphärischen Mediums.

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