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NON-GRAY RADIATIVE TRANSFER IN PLANE-PARALLEL MEDIA WITH REFLECTING BOUNDARIES

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Abstract—A generalized equation of radiative transfer in the two-group picket-fence model is analyzed for a plane parallel, emitting, absorbing and isotropically scattering medium containing uniform heat sources and having boundary surfaces which are diffuse emitters and diffuse reflectors and are maintained at uniform but arbitrary temperatures. The solution of the general problem is expressed by the superposition of simpler problems which are solved by the application of the normal-mode-expansion technique. Highly accurate numerical results are presented for the temperature distribution and the radiative heat flux in the medium.

1. INTRODUCTION

THE EVALUATION of the energy transfer by radiation has been of interest to scientists working with high temperature gases over the years. Early investigations of radiative transfer were mostly in astrophysical applications, and the analyses were based on approximate methods of solution. HOPF[1] appears to be the first investigator to present exact results for several radiative transfer problems. Later, CHANDRASEKHAR,[2] among others, applied a technique based on invariance principles to solve exactly various problems of astrophysics. With the introduction of the normal-mode-expansion technique by CASE,[3] a new era was opened in the field of radiative transfer.

SIEWERT and ZWEIFEL,[4] utilizing this powerful technique, developed the formalism for the exact solution of the equation of radiative transfer in the two-group picket-fence model. Their exact method of solution was applied by SIMMONS and FERZIGER,[5] SIEWERT and ÖZIŞIK,[6] BOND and SIEWERT[7] and REITH *et al.*[8] to radiative transfer problems. Later SIEWERT and SHIEH,[9] SIEWERT and ISHIGURO,[10] REITH and SIEWERT[11] utilized this method and obtained the elementary solutions of the equation of radiative transfer in the two-group picket-fence model with a scattering term included (isotropic and linearly anisotropic scattering) and established the related half-range expansion and orthogonality theorems. YENER *et al.*[12] have applied the method to analyze a generalized form of the equation of radiative transfer in the two-group picket-fence model for a half space.

In the present analysis, radiative heat transfer in an absorbing, emitting and isotropically scattering stagnant gas confined between two emitting and diffusely reflecting parallel boundaries and with uniform internal heat sources is considered in the two-group picket-fence model. The solution of the problem is expressed by the superposition of simpler problems which are in turn solved by the normal-mode-expansion technique.

2. GENERAL FORMULATION

The generalized equation of radiative transfer for one-dimensional plane-parallel, emitting, absorbing, isotropically scattering non-gray media is (CHANDRASEKHAR[2])

$$\mu\frac{\partial}{\partial x}I_\nu(x,\mu)+(\kappa_\nu+\sigma_\nu)I_\nu(x,\mu)=(\kappa_\nu+\epsilon_\nu\sigma_\nu)B_\nu[T(x)]+\frac{1}{2}\sigma_\nu(1-\epsilon_\nu)\int_{-1}^{1}I_\nu(x,\mu')\,\mathrm{d}\mu',\quad(1)$$

where $I_\nu(x,\mu)$ is the spectral radiation intensity, κ_ν and σ_ν are respectively the spectral absorbtion and scattering coefficients. Also, $B_\nu[T(x)]$ is the Planck function at the local temperature $T(x)$ and μ is the direction cosine of the propagating radiation (as measured from the positive x-axis). Here the coefficient $\epsilon_\nu \ll 1$ allows for the possibility that a certain amount of thermal radiation may be associated with the scattering coefficient.

When the energy transfer is by pure radiation (i.e. conductive and convective modes of heat transfer are negligible), the equation of energy conservation in one-dimension for a medium containing heat sources of strength g^* (energy generated/time volume) is given by (ÖZIŞIK[13])

$$\mathrm{d}q^r(x)/\mathrm{d}x = g^*, \tag{2}$$

where $q^r(x)$ is the radiative heat flux, which is related to the radiation intensity $I_\nu(x, \mu)$ by

$$q^r(x) = 2\pi \int_0^\infty \int_{-1}^1 I_\nu(x, \mu')\mu' \,\mathrm{d}\mu' \,\mathrm{d}\nu. \tag{3}$$

Seeking solutions in the medium bounded by parallel planes at $x = 0$ and $x = x_0$, we consider frequency-dependent boundary conditions of the form

$$I_\nu(0, \mu) = \epsilon^*_{1\nu} B_\nu[T_1] + 2\rho^d_{1\nu} \int_0^1 I_\nu(0, -\mu')\mu' \,\mathrm{d}\mu', \; \mu \in (0, 1), \tag{4a}$$

and

$$I_\nu(x_0, -\mu) = \epsilon^*_{2\nu} B_\nu[T_2] + 2\rho^d_{2\nu} \int_0^1 I_\nu(x_0, \mu')\mu' \,\mathrm{d}\mu', \; \mu \in (0, 1). \tag{4b}$$

Here T_α, $\epsilon^*_{\alpha\nu}$ and $\rho^d_{\alpha\nu}$, $\alpha = 1$ or 2, are respectively the temperature, emissivity, and diffuse reflectivity at the boundary surfaces $x = 0$ ($\alpha = 1$) and $x = x_0$ ($\alpha = 2$).

We now assume that the entire frequency spectrum is divided into two regions $\Delta\nu_i$, $i = 1, 2$ in each of which κ_ν, ρ_ν, ϵ_ν, $\epsilon^*_{\alpha\nu}$ and $\rho^d_{\alpha\nu}$ take constant values κ_i, σ_i, ϵ_i, $\epsilon^*_{\alpha i}$, and $\rho^d_{\alpha i}$. Integration of eqn (1) over the $\Delta\nu_i$ yields

$$\mu \frac{\partial}{\partial x} I_i(x, \mu) + (\kappa_i + \sigma_i) I_i(x, \mu) = g_i + \frac{1}{2} \sum_{j=1}^{2} \sigma_{ij} \int_{-1}^1 I_j(x, \mu') \,\mathrm{d}\mu', \tag{5}$$

where

$$\sigma_{ij} = \frac{(\kappa_i + \epsilon_i\sigma_i)(\kappa_j + \epsilon_j\sigma_j)\omega_i}{\sum_{s=1}^{2} (\kappa_s + \epsilon_s\sigma_s)\omega_s} + \delta_{ij}\sigma_j(1 - \epsilon_j) \tag{6}$$

and

$$g_i = \frac{(\kappa_i + \epsilon_i\sigma_i)\omega_i}{4\pi \sum_{i=1}^{2} (\kappa_i + \epsilon_i\sigma_i)\omega_i} g^*. \tag{7}$$

Here

$$I_i(x, \mu) = \int_{\Delta\nu_i} I_\nu(x, \mu) \,\mathrm{d}\nu \tag{8}$$

and

$$\omega_i = [\pi/\bar{\sigma} T^4(x)] \int_{\Delta\nu_i} B_\nu[T(x)] \,\mathrm{d}\nu, \tag{9}$$

where δ_{ij} is the Kronecker delta and $\bar{\sigma}$ is the Stefan–Boltzmann constant. In this analysis ω_1 and $\omega_2 = 1 - \omega_1$ are taken to be constants.

In obtaining eqn (5), we have used the following equation of energy balance:

$$\int_0^\infty (\kappa_\nu + \epsilon_\nu\sigma_\nu)B_\nu[T(x)]\,\mathrm{d}\nu = \frac{g^*}{4\pi} + \frac{1}{2}\int_0^\infty (\kappa_\nu + \epsilon_\nu\sigma_\nu)\int_{-1}^{1} I_\nu(x, \mu')\,\mathrm{d}\mu'\,\mathrm{d}\nu. \tag{10}$$

Introducing an optical variable

$$\mathrm{d}\tau = (\kappa_2 + \sigma_2)\,\mathrm{d}x, \tag{11}$$

we write eqn (5) as

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\tau,\mu) + \mathbf{\Sigma}\mathbf{I}(\tau,\mu) = \mathbf{G} + \mathbf{Q}\int_{-1}^{1}\mathbf{I}(\tau,\mu')\,\mathrm{d}\mu'. \tag{12}$$

Here $\mathbf{I}(\tau, \mu)$ is a two-component vector with elements $I_1(\tau, \mu)$ and $I_2(\tau, \mu)$, while

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}, \tag{13}$$

with

$$\sigma = \frac{\kappa_1 + \sigma_1}{\kappa_2 + \sigma_2}, \qquad \sigma > 1, \tag{14}$$

and

$$\mathbf{G} = \frac{F}{2\sum_{i=1}^{2}\lambda_i\omega_i}\begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \end{bmatrix} \tag{15}$$

with

$$F = \frac{g^*}{2\pi(\kappa_2 + \sigma_2)}. \tag{16}$$

Without loss of generality, σ is taken to be greater than unity and λ_i is defined as

$$\lambda_i = \frac{\kappa_i + \epsilon_i\sigma_i}{\kappa_2 + \sigma_2}. \tag{17}$$

The elements of the 2×2 transfer matrix $\mathbf{Q}$ are

$$q_{ij} = \frac{1}{2}\frac{\sigma_{ij}}{\kappa_2 + \sigma_2}. \tag{18}$$

The required boundary conditions on the vector $\mathbf{I}(\tau, \mu)$ can be obtained by integrating eqns (4) over the frequency band $\Delta\nu_i$, viz.

$$\mathbf{I}(0, \mu) = \mathbf{A}_1 + 2\mathbf{B}_1{}^d\int_0^1 \mathbf{I}(0, -\mu')\mu'\,\mathrm{d}\mu', \qquad \mu \in (0, 1), \tag{19a}$$

and

$$\mathbf{I}(\tau_0, -\mu) = \mathbf{A}_2 + 2\mathbf{B}_2{}^d\int_0^1 \mathbf{I}(\tau_0, \mu')\mu'\,\mathrm{d}\mu', \qquad \mu \in (0, 1). \tag{19b}$$

Here $\tau_0 = (\kappa_2 + \sigma_2)x_0$ is the optical thickness of the slab, and $\mathbf{A}_\alpha$ and $\mathbf{B}_\alpha{}^d$, $\alpha = 1$ or 2, are considered prescribed constants defined in terms of the previously mentioned quantities

$$\mathbf{A}_\alpha = \frac{\bar{\sigma}}{\pi} T_\alpha^4 \begin{bmatrix} \epsilon^*_{\alpha 1}\omega_1 \\ \epsilon^*_{\alpha 2}\omega_2 \end{bmatrix} \tag{20a}$$

and

$$\mathbf{B}_\alpha{}^d = \begin{bmatrix} \rho_{\alpha 1} & 0 \\ 0 & \rho_{\alpha 2} \end{bmatrix}, \qquad \alpha = 1 \text{ or } 2. \tag{20b}$$

Equations (12) and (19) are the basic equations to be solved. To write the desired solution we prefer to use the superposition principle and thus let

$$\mathbf{I}(\tau, \mu) = \mathbf{I}^{(1)}(\tau, \mu) + \mathbf{I}^{(2)}(\tau, \mu). \tag{21}$$

To establish the relation defining $\mathbf{I}^{(1)}(\tau, \mu)$ and $\mathbf{I}^{(2)}(\tau, \mu)$, we consider the following 2×2 *matrix* problems:

$$\mu \frac{\partial}{\partial \tau} \mathbf{\Omega}_i(\tau, \mu) + \mathbf{\Sigma}\mathbf{\Omega}_i(\tau, \mu) = \delta_{2i}\mathbf{I} + \mathbf{Q}\int_{-1}^{1} \mathbf{\Omega}_i(\tau, \mu')\,\mathrm{d}\mu', \qquad i = 1, 2, \tag{22a}$$

$$\mathbf{\Omega}_i(0, \mu) = \delta_{1i}\mathbf{I}, \qquad \mu \in (0, 1), \tag{22b}$$

$$\mathbf{\Omega}_i(\tau_0, -\mu) = \mathbf{0}, \qquad \mu \in (0, 1). \tag{22c}$$

Then it can be shown that $\mathbf{I}^{(1)}(\tau, \mu)$ and $\mathbf{I}^{(2)}(\tau, \mu)$ are given by

$$\mathbf{I}^{(1)}(\tau, \mu) = \mathbf{\Omega}_1(\tau, \mu)\mathbf{L}_1 + \mathbf{\Omega}_1(\tau_0 - \tau, -\mu)\mathbf{R}_1 \tag{23a}$$

and

$$\mathbf{I}^{(2)}(\tau, \mu) = \mathbf{\Omega}_1(\tau, \mu)\mathbf{L}_2 + \mathbf{\Omega}_1(\tau_0 - \tau, -\mu)\mathbf{R}_2 + \mathbf{\Omega}_2(\tau, \mu)\mathbf{G}. \tag{23b}$$

Here the vectors $\mathbf{L}_1$ and $\mathbf{R}_1$ are the solutions of the algebraic equations

$$\mathbf{L}_1 = \mathbf{A}_1 + 2\mathbf{B}_1{}^d[\mathbf{M}\mathbf{L}_1 + \mathbf{N}\mathbf{R}_1] \tag{24a}$$

and

$$\mathbf{R}_1 = \mathbf{A}_2 + 2\mathbf{B}_2{}^d[\mathbf{N}\mathbf{L}_1 + \mathbf{M}\mathbf{R}_1], \tag{24b}$$

and the vectors $\mathbf{L}_2$ and $\mathbf{R}_2$ are the solutions of the following algebraic equations:

$$\mathbf{L}_2 = 2\mathbf{B}_1{}^d[\mathbf{M}\mathbf{L}_2 + \mathbf{N}\mathbf{R}_2 + \mathbf{W}\mathbf{G}] \tag{25a}$$

and

$$\mathbf{R}_2 = 2\mathbf{B}_2{}^d[\mathbf{N}\mathbf{L}_2 + \mathbf{M}\mathbf{R}_2 + \mathbf{W}\mathbf{G}]. \tag{25b}$$

Here **M**, **N**, and **W** are the first moments of the exit distributions of the problems defined by eqns (22), i.e.

$$\mathbf{M} = \int_0^1 \mathbf{\Omega}_1(0, -\mu')\mu'\,\mathrm{d}\mu', \tag{26}$$

$$\mathbf{N} = \int_0^1 \mathbf{\Omega}_1(\tau_0, \mu')\mu'\,\mathrm{d}\mu', \tag{27}$$

and

$$\mathbf{W} = \int_0^1 \mathbf{\Omega}_2(0, -\mu')\mu' \, d\mu' = \int_0^1 \mathbf{\Omega}_2(\tau_0, \mu')\mu' \, d\mu'. \quad (28)$$

We note that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T [\mathbf{M} + \mathbf{N}] = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \quad (29)$$

and

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \mathbf{W} = \tau_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T, \quad (30)$$

and thus not more than six of the eight matrix elements defined by eqns (26) and (27) and two of the four matrix elements defined by eqn (28) are independent. We use the superscripts T and tilde interchangeably to denote the transpose operation.

The temperature distribution in the medium follows from eqn (10):

$$\frac{\bar{\sigma} T^4(\tau)}{\pi} = \frac{1}{2 \sum_{i=1}^{2} \lambda_i \omega_i} \left\{ F + \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \int_{-1}^{1} \mathbf{I}(\tau, \mu') \, d\mu' \right\}. \quad (31)$$

In terms of our basic problems, eqn (31) can be written as

$$T^4(\tau) = [T^{(1)}(\tau)]^4 + [T^{(2)}(\tau)]^4 \quad (32a)$$

where

$$\frac{\bar{\sigma}[T^{(1)}(\tau)]^4}{\pi} = \mathbf{\Gamma}^{(1)}(\tau)\mathbf{L}_1 + \mathbf{\Gamma}^{(1)}(\tau_0 - \tau)\mathbf{R}_1 \quad (32b)$$

and

$$\frac{\bar{\sigma}[T^{(2)}(\tau)]^4}{\pi} = \frac{F}{2 \sum_{i=1}^{2} \lambda_i \omega_i} + \mathbf{\Gamma}^{(1)}(\tau)\mathbf{L}_2 + \mathbf{\Gamma}^{(1)}(\tau_0 - \tau)\mathbf{R}_2 + \mathbf{\Gamma}^{(2)}(\tau)\mathbf{G}. \quad (32c)$$

Here we define

$$\mathbf{\Gamma}^{(1)}(\tau) = \frac{1}{2 \sum_{i=1}^{2} \lambda_i \omega_i} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \int_{-1}^{1} \mathbf{\Omega}_1(\tau, \mu') \, d\mu' \quad (33a)$$

and

$$\mathbf{\Gamma}^{(2)}(\tau) = \frac{1}{2 \sum_{i=1}^{2} \lambda_i \omega_i} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \int_{-1}^{1} \mathbf{\Omega}_2(\tau, \mu') \, d\mu'. \quad (33b)$$

For the two-group model considered, the radiative heat flux, $q^r(\tau)$, given by eqn (3) becomes

$$q^r(\tau) = 2\pi \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \int_{-1}^{1} \mathbf{I}(\tau, \mu')\mu' \, d\mu'. \quad (34)$$

Similarly, in terms of the basic problems, this relation can be written as

$$q^r(\tau) = q^{(1)} + q^{(2)}(\tau),$$

where

$$q^{(1)} = 2\pi \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \mathbf{N}[\mathbf{L}_1 - \mathbf{R}_1]$$

and

$$q^{(2)}(\tau) = (2\tau - \tau_0)\pi F + 2\pi \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \mathbf{N}[\mathbf{L}_2 - \mathbf{R}_2].$$

3. BASIC ANALYSIS

In the previous section we have shown that the solution of the radiative transfer pı given by eqns (12) and (19) could be written in terms of the solutions of the two simpler pr(defined by eqns (22). In this section, following the work of SIEWERT and ISHIGURO,[10] we ţ the solutions of these basic problems.

The transfer matrix $\mathbf{Q}$ in eqn (22a) is, in general, not symmetric and, since we pı symmetric form, we multiply eqn (22a) by

$$\mathbf{P} = \begin{bmatrix} (\omega_2/\omega_1)^{1/2} & 0 \\ 0 & 1 \end{bmatrix}$$

to obtain

$$\mu \frac{\partial}{\partial \tau} \mathbf{\Psi}_i(\tau, \mu) + \mathbf{\Sigma} \mathbf{\Psi}_i(\tau, \mu) = \delta_{2i} \mathbf{P} + \mathbf{C} \int_{-1}^{1} \mathbf{\Psi}_i(\tau, \mu')\, d\mu', \qquad i = 1, 2,$$

with

$$\mathbf{\Psi}_i(0, \mu) = \boldsymbol{\delta}_{1i} \mathbf{P}, \qquad \mu \in (0, 1),$$

and

$$\mathbf{\Psi}_i(\tau_0, -\mu) = \mathbf{0}, \qquad \mu \in (0, 1),$$

where

$$\mathbf{\Psi}_i(\tau, \mu) = \mathbf{P}\mathbf{\Omega}_i(\tau, \mu)$$

and

$$\mathbf{C} = \mathbf{P}\mathbf{Q}\mathbf{P}^{-1}.$$

We note that $\mathbf{C} = \tilde{\mathbf{C}}$ and that the elements of the new transfer matrix $\mathbf{C}$ are given by

$$c_{ij} = \frac{\lambda_i \lambda_j (\omega_i \omega_j)^{1/2}}{2 \sum_{s=1}^{2} \lambda_s \omega_s} + \delta_{ij} \frac{(\sigma \delta_{1i} + \delta_{2i}) - \lambda_i}{2}.$$

Following the work of SIEWERT and SHIEH,[9] for

$$C = \det \mathbf{C} \geq 0$$

and

$$\sigma + 4C - 2c_{11} - 2\sigma c_{22} = 0,$$

we conclude that the dispersion function for this problem has either two zeros at infinity or two zeros at infinity plus two real zeros. Since the latter case is rather special (YENER *et al.*[12]), we consider the case with only two zeros at infinity, i.e.

$$c_{22} > 2C \tanh^{-1}(1/\sigma). \tag{43}$$

A general solution of eqn (37a) can be written as (SIEWERT and ISHIGURO[10])

$$\begin{aligned}\boldsymbol{\Psi}_i(\tau,\mu) = {} & \boldsymbol{\Psi}_p(\tau,\mu) + \boldsymbol{\Phi}_+\tilde{\mathbf{A}}_{i+} + \boldsymbol{\Phi}_-(\tau,\mu)\tilde{\mathbf{A}}_{i-} \\ & + \int_0^{1/\sigma} [\boldsymbol{\Phi}_1^{(1)}(\eta,\mu)\tilde{\mathbf{A}}_{i1}^{(1)}(\eta) + \boldsymbol{\Phi}_2^{(1)}(\eta,\mu)\tilde{\mathbf{A}}_{i2}^{(1)}(\eta)]\,e^{-\tau/\eta}\,d\eta \\ & + \int_0^{1/\sigma} [\boldsymbol{\Phi}_1^{(1)}(-\eta,\mu)\tilde{\mathbf{A}}_{i1}^{(1)}(-\eta) + \boldsymbol{\Phi}_2^{(1)}(-\eta,\mu)\tilde{\mathbf{A}}_{i2}^{(1)}(-\eta)]\,e^{\tau/\eta}\,d\eta \\ & + \int_{1/\sigma}^{1} \boldsymbol{\Phi}^{(2)}(\eta,\mu)\tilde{\mathbf{A}}_i^{(2)}(\eta)\,e^{-\tau/\eta}\,d\eta \\ & + \int_{1/\sigma}^{1} \boldsymbol{\Phi}^{(2)}(-\eta,\mu)\tilde{\mathbf{A}}_i^{(2)}(-\eta)\,e^{\tau/\eta}\,d\eta,\end{aligned} \tag{44}$$

where $\boldsymbol{\Phi}_1^{(1)}(\eta,\mu)$, $\boldsymbol{\Phi}_2^{(1)}(\eta,\mu)$, and $\boldsymbol{\Phi}^{(2)}(\eta,\mu)$ are the continuum eigensolutions (SIEWERT and ISHIGURO[10]) and $\boldsymbol{\Phi}_+$ and $\boldsymbol{\Phi}_-(\tau,\mu)$ are the discrete eigensolutions (YENER *et al.*[12]); $\mathbf{A}_{i+}$, $\mathbf{A}_{i-}$, $\mathbf{A}_{i1}^{(1)}(\mp\eta)$, $\mathbf{A}_{i2}^{(1)}(\mp\eta)$, and $\mathbf{A}_i^{(2)}(\mp\eta)$ are the expansion vector coefficients to be determined; $\boldsymbol{\Psi}_p(\tau,\mu)$ is the particular solution corresponding to the inhomogeneous term $\delta_{2i}\mathbf{P}$ and is written as

$$\boldsymbol{\Psi}_p(\tau,\mu) = [\tau^2\mathbf{I} - 2\tau\mu\boldsymbol{\Sigma}^{-1} + 2\mu^2\boldsymbol{\Sigma}^{-2}]\boldsymbol{\Phi}_+\tilde{\boldsymbol{\alpha}} + \mathbf{K}\tilde{\boldsymbol{\beta}}, \tag{45a}$$

where

$$\mathbf{K} = \begin{bmatrix} c_{12} \\ c_{22} - \frac{1}{2} \end{bmatrix}, \tag{45b}$$

$$\boldsymbol{\alpha} = -\frac{3\delta_{2i}}{4p}\mathbf{P}\boldsymbol{\Phi}_+, \tag{45c}$$

and

$$\boldsymbol{\beta} = \frac{\delta_{2i}}{\lambda p r}\mathbf{P}[p\mathbf{K} - s\boldsymbol{\Phi}_+], \tag{45d}$$

with

$$r = \tilde{\mathbf{K}}\mathbf{K}, \tag{45e}$$

$$s = \tilde{\mathbf{K}}\mathbf{C}\boldsymbol{\Sigma}^{-2}\boldsymbol{\Phi}_+, \tag{45f}$$

$$p = \tilde{\boldsymbol{\Phi}}_+\mathbf{C}\boldsymbol{\Sigma}^{-2}\boldsymbol{\Phi}_+, \tag{45g}$$

and

$$\lambda = \sigma + 1 - 2(c_{11} + c_{22}). \tag{45h}$$

If we now substitute the solution given by eqn (44) into equations (37b) and (37c) we get

$$\begin{aligned}\mathbf{L}_{i1}(\mu) = {} & \boldsymbol{\Phi}_+\tilde{\mathbf{A}}_{i+} + \int_0^{1/\sigma} [\boldsymbol{\Phi}_1^{(1)}(\eta,\mu)\tilde{\mathbf{A}}_{i1}^{(1)}(\eta) + \boldsymbol{\Phi}_2^{(1)}(\eta,\mu)\tilde{\mathbf{A}}_{i2}^{(1)}(\eta)]\,d\eta \\ & + \int_{1/\sigma}^{1} \boldsymbol{\Phi}^{(2)}(\eta,\mu)\tilde{\mathbf{A}}_i^{(2)}(\eta)\,d\eta, \qquad \eta \in (0,1),\end{aligned} \tag{46a}$$

and

$$\mathbf{L}_{i2}(\mu)=\mathbf{\Phi}_{+}\tilde{\mathbf{A}}_{i+}+\int_{0}^{1/\sigma}[\mathbf{\Phi}_{1}^{(1)}(\eta,\mu)\tilde{\mathbf{A}}_{i1}^{(1)}(-\eta)+\mathbf{\Phi}_{2}^{(1)}(\eta,\mu)\tilde{\mathbf{A}}_{i2}^{(1)}(-\eta)]\,e^{\tau_0/\eta}\,d\eta$$

$$+\int_{1/\sigma}^{1}\mathbf{\Phi}^{(2)}(\eta,\mu)\tilde{\mathbf{A}}_{i}^{(2)}(-\eta)\,e^{\tau_0/\eta}\,d\eta,\qquad \mu\in(0,1),\tag{46b}$$

where

$$\mathbf{L}_{i1}(\mu)=\delta_{1i}\mathbf{P}-\mathbf{\Psi}_{p}(0,\mu)-\mathbf{\Phi}_{-}(0,\mu)\tilde{\mathbf{A}}_{i-}$$

$$-\int_{0}^{1/\sigma}[\mathbf{\Phi}_{1}^{(1)}(-\eta,\mu)\tilde{\mathbf{A}}_{i1}^{(1)}(-\eta)+\mathbf{\Phi}_{2}^{(1)}(-\eta,\mu)\tilde{\mathbf{A}}_{i2}^{(1)}(-\eta)]\,d\eta$$

$$-\int_{1/\sigma}^{1}\mathbf{\Phi}^{(2)}(-\eta,\mu)\tilde{\mathbf{A}}_{i}^{(2)}(-\eta)\,d\eta,\qquad \mu\in(0,1),\tag{47a}$$

and

$$\mathbf{L}_{i2}(\mu)=-\mathbf{\Psi}_{p}(\tau_0,-\mu)-\mathbf{\Phi}_{-}(\tau_0,-\mu)\tilde{\mathbf{A}}_{i-}$$

$$-\int_{0}^{1/\sigma}[\mathbf{\Phi}_{1}^{(1)}(-\eta,\mu)\tilde{\mathbf{A}}_{i1}^{(1)}(\eta)+\mathbf{\Phi}_{2}^{(1)}(-\eta,\mu)\tilde{\mathbf{A}}_{i2}^{(1)}(\eta)]\,e^{-\tau_0/\eta}\,d\eta$$

$$-\int_{1/\sigma}^{1}\mathbf{\Phi}^{(2)}(-\eta,\mu)\tilde{\mathbf{A}}_{i}^{(2)}(\eta)\,e^{-\tau_0/\eta}\,d\eta,\qquad \mu\in(0,1).\tag{47b}$$

Equations (46) and (47) are two coupled singular integral equations. They may, however, be transformed to coupled regular integral equations by making use of the orthogonality properties of the eigenfunctions. Since $\mathbf{L}_{i1}(\mu)$ and $\mathbf{L}_{i2}(\mu)$ are themselves expressed in terms of expansion coefficients, eight coupled integral equations for the eight unknown expansion vector coefficients are obtained rather than closed-form results; since the final equations are long, we do not present them here but note that they can be found elsewhere (YENER[14]). These coupled integral equations were solved numerically and highly accurate numerical results were obtained for the expansion coefficients. Once the expansion coefficients are known various other quantities are readily computed from definitions given previously.

The angular exit distributions $\mathbf{\Psi}_i(0,-\mu)$, $\mu\in(0,1)$ and $i=1$ and 2, follow from eqn (44) by setting $\tau=0$ and considering $\mu<0$. Similarly, $\mathbf{\Psi}_i(\tau_0,\mu)$, $\mu\in(0,1)$ and $i=1$ and 2, follows from the same equation by setting $\tau=\tau_0$ and considering $\mu>0$. First moments of these exit distributions are the basic quantities required for the calculation of the matrices **M**, **N** and **W**. These moments can be obtained by multiplying the exit distributions by μ and then integrating over μ from 0 to 1 as follows:

$$\int_{0}^{1}\mathbf{\Psi}_i(0,-\mu')\mu'\,d\mu'=\frac{1}{2}[\mathbf{\Sigma}^{-2}\mathbf{\Phi}_{+}\tilde{\boldsymbol{\alpha}}+\mathbf{K}\boldsymbol{\beta}]+\frac{1}{2}\mathbf{\Phi}_{+}\tilde{\mathbf{A}}_{i+}+\frac{1}{3}\mathbf{\Sigma}^{-1}\mathbf{\Phi}_{i+}\tilde{\mathbf{A}}_{i-}$$

$$+\int_{0}^{1}[\mathbf{L}(-\eta)\tilde{\mathbf{A}}_i(\eta)+\mathbf{L}(\eta)\tilde{\mathbf{A}}_i(-\eta)]\,d\eta\tag{48a}$$

and

$$\int_{0}^{1}\mathbf{\Psi}_i(\tau_0,\mu')\mu'\,d\mu'=\left[\frac{1}{2}\tau_0^{2}-\frac{2}{3}\tau_0\mathbf{\Sigma}^{-1}+\frac{1}{2}\mathbf{\Sigma}^{-2}\right]\mathbf{\Phi}_{+}\tilde{\boldsymbol{\alpha}}+\frac{1}{2}\mathbf{K}\boldsymbol{\beta}$$

$$+\frac{1}{2}\mathbf{\Phi}_{+}\tilde{\mathbf{A}}_{i+}+\left[\frac{1}{2}\tau_0-\frac{1}{3}\mathbf{\Sigma}^{-1}\right]\mathbf{\Phi}_{i+}\mathbf{A}_{i-}$$

$$+\int_{0}^{1}[\mathbf{L}(\eta)\tilde{\mathbf{A}}_i(\eta)\,e^{-\tau_0/\eta}+\mathbf{L}(-\eta)\tilde{\mathbf{A}}_i(-\eta)\,e^{\tau_0/\eta}]\,d\eta,\tag{48b}$$

where we have defined, for $\eta \geq 0$,

$$\mathbf{L}(-\eta) = \eta\left\{\mathbf{I} + \eta\boldsymbol{\Sigma}\begin{bmatrix} \ln\dfrac{\sigma\eta}{1+\sigma\eta} & 0 \\ 0 & \ln\dfrac{\eta}{1+\eta}\end{bmatrix}\right\}\mathbf{C} \tag{49a}$$

and

$$\mathbf{L}(\eta) = \eta(\boldsymbol{\Sigma} - 2\mathbf{C}) + \mathbf{L}(-\eta). \tag{49b}$$

Here it should be noted that

$$\int_0^1 \boldsymbol{\Omega}_i(0, -\mu')\mu'\,\mathrm{d}\mu' = \mathbf{P}^{-1}\int_0^1 \boldsymbol{\Psi}_i(0, -\mu')\mu'\,\mathrm{d}\mu' \tag{50a}$$

and

$$\int_0^1 \boldsymbol{\Omega}_i(\tau_0, \mu')\mu'\,\mathrm{d}\mu' = \mathbf{P}^{-1}\int_0^1 \boldsymbol{\Psi}_i(\tau_0, \mu')\mu'\,\mathrm{d}\mu'. \tag{50b}$$

Alternatively, we also calculated the first moments of $\boldsymbol{\Psi}_i(0, -\mu)$ and $\boldsymbol{\Psi}_i(\tau_0, \mu)$, $\mu \in (0, 1)$ and $i = 1$ and 2, using the **S** matrix defined by SIEWERT and ISHIGURO[10]. Since the resulting equations are lengthy, we do not present them here. The matrices **M**, **N** and **W** were computed using the results from both approaches and the numbers we obtained from both methods agreed at least to eight significant figures.

4. NUMERICAL ANALYSIS AND RESULTS

The temperature distribution and the heat flux are the two physical quantities of practical interest. As discussed previously, the solution of the problem has been related to the solutions of two basic problems defined by eqns (22). To perform the calculations numerically, the integral region [0, 1] was divided into a number of subintervals and a 40-point Gaussian quadrature scheme was used in each of these subintervals. For almost all of the cases considered here, six subintervals provided results of sufficient accuracy. All calculations were performed in double precision arithmetic on an IBM 360/175 computer. The iterative procedure was terminated when successive iterates yielded expansion coefficients in agreement to at least eight significant figures.

Once the expansion coefficients are determined, the matrices **M**, **N** and **W** defined by eqns (26)–(28) are readily evaluated by utilizing the expressions given by eqns (48) and (50). We list in Table 1 a choice of eight elements of these matrices for representative values of the parameters σ, λ_1, λ_2, ω_1 and τ_0. A check on the computed values of matrices **M**, **N** and **W** is provided by verifying eqns (29) and (30) for the cases considered here.

It is clear from eqns (32) and (35) that the "temperature functions" $\boldsymbol{\Gamma}^{(1)}(\tau)$ and $\boldsymbol{\Gamma}^{(2)}(\tau)$ and the vectors $\mathbf{L}_i$ and $\mathbf{R}_i$, $i = 1$ and 2, are needed to evaluate the temperature distribution and the net radiative flux in the slab. The functions $\boldsymbol{\Gamma}^{(1)}(\tau)$ and $\boldsymbol{\Gamma}^{(2)}(\tau)$ are determined from eqns (33) for any given values of the parameters. The vectors $\mathbf{L}_i$ and $\mathbf{R}_i$, $i = 1$ and 2, are evaluated from the solution of eqns (24) and (25) once the reflectivities and emissivities of the boundary surfaces of the slab are specified.

In Tables 2 and 3, the elements of the functions $\boldsymbol{\Gamma}^{(1)}(\tau)$ and $\boldsymbol{\Gamma}^{(2)}(\tau)$ are presented for typical

Table 1. Eight elements of **M**, **N** and **W** with $\tau_0 = 1{\cdot}0$

σ	λ_1	λ_2	ω_1	M_{11}	M_{12}	M_{21}	N_{12}	N_{21}	N_{22}	W_{11}	W_{12}
5	2.5	0.5	0.2	0.18208	0.04713	0.18853	0.02656	0.10626	0.24620	0.37581	0.11050
5	2.5	0.5	0.4	0.25498	0.08377	0.12566	0.05052	0.07577	0.21981	0.55470	0.20555
5	2.5	0.5	0.6	0.31093	0.11476	0.07651	0.07275	0.04850	0.19588	0.71593	0.29090
5	2.5	0.5	0.8	0.35689	0.14201	0.03550	0.09356	0.02339	0.17383	0.86359	0.35890
5	1	1	0.2	0.21050	0.04121	0.16485	0.02449	0.09795	0.24901	0.42829	0.09979
10	1	1	0.2	0.27141	0.03582	0.14330	0.01894	0.07575	0.25402	0.35468	0.08087

Table 2. Temperature function $\Gamma^{(1)}(\tau)$ with $\tau_0 = 1{\cdot}0$

σ	λ_1	λ_2	ω_1	τ: 0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
				$\Gamma_1^{(1)}(\tau)$										
5	2.5	0.5	0.2	2.0598	1.3900	1.0307	1.7892	0.6170	0.4893	0.3908	0.3113	0.2437	0.1814	0.1116
5	2.5	0.5	0.4	1.5388	1.639	0.9313	0.7560	0.6176	0.5047	0.4097	0.3273	0.2527	0.1815	0.1010
5	2.5	0.5	0.6	1.2411	1.0116	0.8521	0.7206	0.6077	0.5084	0.4191	0.3369	0.2592	0.1829	0.0962
5	2.5	0.5	0.8	1.0466	0.9009	0.7882	0.6871	0.5937	0.5060	0.4224	0.3419	0.2630	0.1838	0.0933
5	1	1	0.2	0.9285	0.7446	0.6178	0.5178	0.4363	0.3682	0.3099	0.2585	0.2115	0.1664	0.1162
10	1	1	0.2	0.9671	0.6979	0.5294	0.4124	0.3287	0.2669	0.2191	0.1802	0.1463	0.1145	0.0792
σ	λ_1	λ_2	ω_1	$\Gamma_2^{(1)}(\tau)$										
5	2.5	0.5	0.2	0.5215	0.5787	0.5848	0.5695	0.5408	0.5027	0.4573	0.4054	0.3466	0.2784	0.1856
5	2.5	0.5	0.4	0.4134	0.5075	0.5398	0.5435	0.5276	0.4969	0.4541	0.4009	0.3375	0.2622	0.1601
5	2.5	0.5	0.6	0.3494	0.4579	0.5041	0.5195	0.5124	0.4873	0.4473	0.3942	0.3289	0.2504	0.1448
5	2.5	0.5	0.8	0.3062	0.4207	0.4748	0.4978	0.4968	0.4761	0.4386	0.3862	0.3204	0.2406	0.1342
5	1	1	0.2	0.7522	0.7098	0.6691	0.6262	0.5808	0.5329	0.4827	0.4297	0.3736	0.3124	0.2366
10	1	1	0.2	0.7522	0.7262	0.6932	0.6531	0.6078	0.5583	0.5053	0.4487	0.3879	0.3207	0.2363

Table 3. Temperature function $\Gamma^{(2)}(\tau)$ with $\tau_0 = 1{\cdot}0$

σ	λ_1	λ_2	ω_1	τ: 0.0	0.1	0.2	0.3	0.4	0.5
				$\Gamma_1^{(2)}(\tau)$					
5	2.5	0.5	0.2	1.1195	1.7777	2.1501	2.3873	2.5211	2.5644
5	2.5	0.5	0.4	1.1129	1.9457	2.4367	2.7569	2.9402	3.0000
5	2.5	0.5	0.6	1.0557	1.9507	2.4993	2.8656	3.0782	3.1481
5	2.5	0.5	0.8	0.9894	1.8864	2.4532	2.8384	3.0646	3.1393
5	1	1	0.2	1.0027	1.5525	1.8801	2.0944	2.2172	2.2573
10	1	1	0.2	1.0011	1.7151	2.1368	2.4046	2.5544	2.6027
σ	λ_1	λ_2	ω_1	$\Gamma_2^{(2)}(\tau)$					
5	2.5	0.5	0.2	0.9825	1.3818	1.6175	1.7719	1.8608	1.8899
5	2.5	0.5	0.4	0.8228	1.2662	1.5388	1.7213	1.8276	1.8626
5	2.5	0.5	0.6	0.6877	1.1404	1.4291	1.6264	1.7426	1.7811
5	2.5	0.5	0.8	0.5926	1.0372	1.3288	1.5312	1.6516	1.6916
5	1	1	0.2	1.0586	1.4412	1.6667	1.8147	1.8999	1.9278
10	1	1	0.2	1.0590	1.4806	1.7313	1.8953	1.9893	2.0110

Table 4. Data for selected cases with $T_1/T_2 = 2{\cdot}0$ and $\tau_0 = 1{\cdot}0$

Case	σ	λ_1	λ_2	ω_1	ρ_{11}^d	ρ_{12}^d	ρ_{21}^d	ρ_{22}^d	ε_{11}^*	ε_{12}^*	ε_{21}^*	ε_{22}^*
I	5	2.5	0.5	0.2	0.8	0.9	0.7	0.8	0.2	0.1	0.3	0.2
II	5	2.5	0.5	0.2	0.1	0.3	0.2	0.1	0.9	0.7	0.8	0.9
III	5	2.5	0.5	0.6	0.8	0.9	0.7	0.8	0.2	0.1	0.3	0.2
IV	5	2.5	0.5	0.6	0.1	0.3	0.2	0.1	0.9	0.7	0.7	0.9
V	5	1	1	0.2	0.8	0.9	0.7	0.8	0.2	0.1	0.3	0.2
VI	5	1	1	0.2	0.1	0.3	0.2	0.1	0.9	0.7	0.8	0.9
VII	10	1	1	0.2	0.8	0.9	0.7	0.8	0.2	0.1	0.3	0.2
VIII	10	1	1	0.2	0.1	0.3	0.2	0.1	0.9	0.7	0.8	0.9

Table 5. The temperature distribution $T^{(1)}(\tau)$ and heat flux $q^{(1)}$ for selected cases

τ (Case)	$\left[\frac{T^{(1)}(\tau)}{T_1}\right]^4$ 0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	$q^{(1)}/\bar{\sigma}T_1^4$
I	0.47799	0.45076	0.43335	0.41934	0.40715	0.39586	0.38481	0.37339	0.36083	0.34590	0.32365	0.07016
II	0.76638	0.68423	0.62586	0.57533	0.52898	0.48484	0.44149	9.39769	0.35195	0.30174	0.23609	0.35212
III	0.56825	0.53163	0.50400	0.47897	0.45550	0.43286	0.41047	0.38776	0.36406	0.33828	0.30539	0.07910
IV	0.85722	0.76970	0.70024	0.63630	0.57545	0.51637	0.45809	0.39966	0.33999	0.27711	0.20089	0.27523
V	0.44780	0.43381	0.42289	0.41303	0.40377	0.39482	0.38596	0.37698	0.36761	0.35744	0.34469	0.06984
VI	0.70535	0.64895	0.60340	0.56118	0.52072	0.48115	0.44185	0.40221	0.36147	0.31818	0.26548	0.36109
VII	0.45865	0.44001	0.42647	0.41512	0.40504	0.39560	0.38632	0.37675	0.36637	0.35440	0.33847	0.06771
VIII	0.71302	0.65090	0.60223	0.55847	0.51742	0.47774	0.43844	0.39857	0.35703	0.31199	0.25610	0.34766

Table 6. The temperature distribution $T^{(2)}(\tau/\tau_0)$ and heat flux $q^{(2)}(\tau_0/2)$ for selected cases

τ (Case)	$\frac{\bar{\sigma}}{\pi F}\left[T^{(2)}(\tau)\right]^4$ 0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	$\frac{q^{(2)}(\tau_0/2)}{\pi F}$
I	6.03825	6.30048	6.43597	6.51020	6.53663	6.52009	6.46100	6.35597	6.19638	5.96202	5.55932	0.28896
II	1.29453	1.54804	1.69023	1.77921	1.82708	1.83884	1.81626	1.75873	1.66271	1.51845	1.27550	0.09944
III	4.78144	5.15613	5.37229	5.50154	5.55611	5.54014	5.45407	5.29531	5.05708	4.72253	4.19792	0.20831
IV	0.95106	1.37733	1.63976	1.81622	1.92050	1.95796	1.93042	1.83699	1.67341	1.42751	1.02726	0.02665
V	6.14124	6.33008	6.43219	6.48880	6.50784	6.49214	6.44211	6.35617	6.22995	6.05284	5.77115	0.29015
VI	1.30123	1.50255	1.61875	1.69204	1.73057	1.73744	1.71357	1.75815	1.76811	1.43512	1.21505	0.10420
VII	6.20656	6.45603	6.59351	6.67077	6.70014	6.68674	6.63141	6.53113	6.37809	6.15569	5.79679	0.28761
VIII	1.30550	1.54306	1.68057	1.76604	1.81035	1.81819	1.79106	1.72782	1.72412	1.46939	1.21434	0.11006

values of the parameters. Because of the symmetry of the elements of $\Gamma^{(2)}(\tau)$ [i.e. $\Gamma^{(2)}(\tau) = \Gamma^{(2)}(\tau_0 - \tau)$], the elements of this function are tabulated for only half the slab thickness. This symmetry condition has been used as one of the checks on the accuracy of the numerical results; the results agreed at least up to ten significant figures for the cases presented here. In Tables 5 and 6, we present the temperature distribution and the heat flux in the medium for the cases listed in Table 4.

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