

## TEMPERATURE INVERSION IN THE KINETIC THEORY OF EVAPORATION

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## ABSTRACT

The singular-eigenfunction-expansion technique is applied to the problem of evaporation and condensation between parallel surfaces. "Critical" values of  $\beta$ , the slope of the saturated vapor density-temperature curve, are reported so that the condition for which the center-line slope of the temperature profile in the vapor will be in opposition to the imposed temperature gradient can readily be noted.

## INTRODUCTION

A problem of recent interest to workers in the kinetic theory of gases is that of evaporation and condensation. Y. P. Pao [1,2], using mathematical tools provided by the Wiener-Hopf technique and the linearized single-relaxation model of the Boltzmann equation, considered the evaporation problems in a half space and between parallel surfaces. Siewert and Thomas [3] subsequently showed Pao's result [1] to be in error, and calculated very accurate results for the half-space problem through application of the singular-eigenfunction-expansion technique [4] to the same modeled Boltzmann equation. Quite recently, Cipolla, Lang, and Loyalka [5] used variation methods to achieve results in essential agreement with those of Siewert and Thomas [3].

The present work is concerned with an application of the analysis of Kriese, Chang, and Siewert [4], used in Refs. [3], [6], and [7], to the parallel-surfaces evaporation problem considered by Pao in Ref. [2]. Using this method, we have been able to confirm a rather novel prediction, made by Pao [2], concerning the shape of the temperature profile in the vapor, and to give explicit numerical values for the relevant results.

## FORMULATION OF THE PROBLEM

We consider the problem of a vapor between two interphase (vapor-liquid or vapor-solid) surfaces maintained at  $x=\pm d/2$ ; we assume that the condensed phase in  $x<-d/2$  is kept at temperature

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$T_0 - 1/2\Delta T$  and that the condensed phase in  $x > d/2$  is kept at temperature  $T_0 + 1/2\Delta T$ . As a result of evaporation and condensation there is a flow of mass and energy from  $x = -d/2$  to  $x = d/2$ . Within the vapor, we assume the state of the fluid to be described by the linearized single-relaxation model of the Boltzmann equation:

$$c_x \frac{\partial}{\partial x} f(x, \tilde{c}) + f(x, \tilde{c}) = \pi^{-3/2} [N(x) + (c^2 - \frac{3}{2})T(x) + 2c_x U(x)]. \quad (1)$$

Here  $f(x, \tilde{c})$  is the perturbation of the particle-distribution function,  $x$  is the spatial variable,  $c$  is the molecular velocity, and  $c_x$  is the  $x$ -component of the  $\tilde{c}$  velocity, all in dimensionless units. In addition,  $N(x)$ ,  $T(x)$ , and  $U(x)$  represent perturbations of the number density, the temperature, and the  $x$ -component of mass flow:

$$N(x) = \int f(x, \tilde{c}) \exp(-c^2) d^3c, \quad (2a)$$

$$T(x) = \frac{2}{3} \int f(x, \tilde{c}) (c^2 - \frac{3}{2}) \exp(-c^2) d^3c, \quad (2b)$$

and

$$U(x) = \int f(x, \tilde{c}) c_x \exp(-c^2) d^3c. \quad (2c)$$

Since it follows from Eqs. (1) and (2c) that  $U(x)$  is a constant, we find it convenient to introduce

$$h(x, \tilde{c}) = f(x, \tilde{c}) - (2/\pi^{3/2}) c_x U, \quad (3)$$

and subsequently consider the equation

$$c_x \frac{\partial}{\partial x} h(x, \tilde{c}) + h(x, \tilde{c}) = \pi^{-3/2} [N(x) + (c^2 - \frac{3}{2})T(x)]. \quad (4)$$

We are interested here only in density and temperature effects, so we may decompose the equation in the manner discussed by Cercignani [9] to obtain

$$\mu \frac{\partial}{\partial x} \tilde{\Psi}(x, \mu) + \tilde{\Psi}(x, \mu) = \pi^{-1/2} Q(\mu) \int_{-\infty}^{\infty} \tilde{Q}(\mu') \tilde{\Psi}(x, \mu') \exp(-\mu'^2) d\mu', \quad (5)$$

where  $Q(\mu)$  is a matrix of polynomials [4]. The variable  $\mu$  now represents the  $x$ -component of the velocity, and  $\tilde{\Psi}(x, \mu)$  is a two-vector simply related to the temperature and density of the gas:

$$N(x) = \pi^{1/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \int_{-\infty}^{\infty} \tilde{\Psi}(x, \mu) \exp(-\mu^2) d\mu \quad (6)$$

and

$$T(x) = \frac{2}{3} \pi^{1/2} \int_{-\infty}^{\infty} \begin{bmatrix} \mu^2 - \frac{1}{2} \\ 1 \end{bmatrix}^T \tilde{\Psi}(x, \mu) \exp(-\mu^2) d\mu. \quad (7)$$

We use the superscripts T and tilde interchangeably to denote the transpose operation.

At the two interphase surfaces, we assume that the vapor molecules striking the surface are absorbed and re-emitted with a Maxwellian distribution of velocities characterized by the temperature at the respective surface. Hence the boundary conditions are

$$f(\mp \frac{\delta}{2}, c) = \mp \frac{1}{2} [\Delta N + (c^2 - \frac{3}{2}) \Delta T], \quad c_x > 0, \quad (8)$$

where  $\delta$  is a nondimensional distance and  $\Delta N = \beta \Delta T$ , with  $\beta$  the slope of the saturated vapor density-temperature curve. Couched in terms of the vector  $\tilde{\Psi}(x, \mu)$ , we find this boundary condition to be

$$\tilde{\Psi}(\mp \frac{\delta}{2}, \pm \mu) = \mp \sqrt{\pi} \begin{bmatrix} \mu^2 - \frac{1}{2} + \beta + 2\pi^{-3/2} U \mu \\ 1 \end{bmatrix}, \quad \mu > 0. \quad (9)$$

In Eq. (9), for notational convenience we have absorbed into U and  $\tilde{\Psi}$  a factor  $\Delta T$ .

From an asymptotic solution for large  $\delta$ , Pao [2] has predicted that for certain values of  $\beta$  the slope of the temperature profile in the main body of the vapor will be in opposition to the imposed temperature gradient. It is our purpose here to investigate this prediction by solving for the temperature profile using the singular-eigenfunction-expansion technique, developed for Eq. (5) by Kriese, Chang, and Siewert [4].

#### SOLUTION OF THE PROBLEM

Kriese, Chang, and Siewert [4] have expressed a general solution of Eq. (5) as

$$\begin{aligned} \tilde{\Psi}(x, \mu) = & \sum_{\alpha=1}^2 A_{\alpha} \tilde{\Phi}_{\alpha}(\mu) + \sum_{\alpha=3}^4 A_{\alpha} \tilde{\Psi}_{\alpha}(x, \mu) \\ & + \sum_{\alpha=1}^2 \int_{-\infty}^{\infty} A_{\alpha}(\eta) \tilde{\Phi}_{\alpha}(\eta, \mu) e^{-x/\eta} d\eta, \end{aligned} \quad (10)$$

where the  $\tilde{\Phi}_{\alpha}(\mu)$ ,  $\tilde{\Psi}_{\alpha}(x, \mu)$ , and  $\tilde{\Phi}_{\alpha}(\eta, \mu)$  are the normal modes given explicitly in Ref. [4], and the  $\tilde{A}_{\alpha}$ ,  $\alpha=1, \dots, 4$  and  $A_{\alpha}(\eta)$ ,  $\alpha=1$  and 2 are expansion coefficients. Substituting this general solution into Eq. (7) for the temperature perturbation, we find

$$T(x) = -\sqrt{\frac{2}{3\pi}} A_3 x - \frac{2}{\pi} \int_0^{\infty} A_1(\eta) \sinh(x/\eta) e^{-\eta^2} d\eta, \quad (11)$$

so that it is clear that we must determine the expansion coefficients  $A_3$  and  $A_1(\eta)$  in order to establish the temperature profile. Using the solution (10) in the boundary conditions given by Eq. (9), we find a coupled pair of singular integral equations for the expansion coefficients. We then may apply the half-range ( $\mu > 0$ ) orthogonality theorem of Kriese, Chang, and Siewert [4] to convert these singular integral equations into a coupled pair of Fredholm integral equations. We have found that iterative solutions of these Fredholm equations converge very rapidly, so that solutions for the expansion coefficients may be straightforwardly obtained, with great precision.

In a companion paper [8] we report our complete results for the temperature and density profiles, and the mass and heat flux. We did indeed find the temperature profile to be in opposition to the imposed temperature gradient for certain values of  $\beta$ , which are dependent on the inverse Knudsen number  $\delta$ . In fact, for all values of  $\beta$  greater than a "critical" value,  $\beta_c(\delta)$ , the computed temperature profile was in opposition to the imposed gradient. We should like to report here our computed values for  $\beta_c(\delta)$ . We define  $\beta_c(\delta)$  as that  $\beta$  for which

$$\left. \frac{d}{dx} T(x) \right|_{x=0} = 0. \quad (12)$$

By solving the coupled Fredholm integral equations mentioned above in conjunction with the constraint given by Eq. (12), we were able to compute  $\beta_c(\delta)$  quite straightforwardly.

## RESULTS AND CONCLUSIONS

The accompanying table gives our computed results for  $\beta_c(\delta)$ . It should be noted that  $\beta_c(\delta)$  soon approaches an

asymptotic value, as  $\delta$  increases, of 3.7723..., to be compared with Pao's value of 3.5.

We hope that these computed results might help stimulate an experimental investigation of this unusual phenomenon.

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TABLE I

COMPUTED VALUES OF  $\beta_c$  VERSUS INVERSE KNUDSEN NUMBER,  $\delta$ 

$\delta$	$\beta_c$
2.0	4.3597
4.0	4.1470
6.0	3.9997
8.0	3.9108
10.0	3.8576
20.0	3.7809
200.0	3.7723
2000.0	3.7723