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ON AN EXACT SOLUTION IN THE THEORY OF LINE FORMATION IN STELLAR ATMOSPHERES

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ABSTRACT

A rigorous solution for the monochromatic radiation intensity at any optical distance into a semi-infinite stellar medium is presented. The model of the atmosphere allows for general continuous absorption as well as for monochromatic line scattering that is linearly anisotropic in the cosine of the scattering angle; however, it is assumed that the Planck distribution can be represented by a linear function of the optical thickness.

I. INTRODUCTION

The equation of radiative transfer for a stellar atmosphere in which monochromatic scattering and local thermodynamic sources are present can be written in the form (Chandrasekhar 1950)

$$\frac{\mu}{\rho}\frac{\partial}{\partial z} I_\nu(z,\mu) + (\kappa_\nu + \sigma_\nu) I_\nu(z,\mu) = \tfrac{1}{2}\sigma_\nu(1-\epsilon_\nu)\int_{-1}^{1} p(\mu,\mu') I_\nu(z,\mu')\, d\mu' + (\kappa_\nu + \epsilon_\nu\sigma_\nu) B_\nu[T(z)], \tag{1}$$

where $p(\mu,\mu')$ is the phase function for conservative, linearly anisotropic scattering,

$$p(\mu,\mu') = 1 + \varpi_1 \mu\mu' . \tag{2}$$

The specific intensity within an absorption line, I_ν, is a function of the depth, z, of the atmosphere and of μ, the cosine of the angle denoting the inclination to the *inward* normal. The absorption and scattering coefficients are κ_ν and σ_ν, respectively. Also, ρ is the density of the medium, $B_\nu[T(z)]$ is the Planck black-body function,

$$B_\nu[T(z)] = \frac{2h\nu^3}{c^2}\left[\exp\frac{h\nu}{kT(z)} - 1\right]^{-1}, \tag{3}$$

and $T(z)$ is the local temperature. The parameter ϵ_ν is included to allow for the possibility of having thermal emission associated with the scattering coefficient.

In terms of the optical thickness,

$$t_\nu = \int_0^z \rho(\kappa_\nu + \sigma_\nu)\, dz , \tag{4}$$

equation (1) takes a more tractable form:

$$\mu \frac{\partial}{\partial t_\nu} I_\nu(t_\nu,\mu) + I_\nu(t_\nu,\mu) = \tfrac{1}{2}\int_{-1}^{1} (c_{\nu 0} + c_{\nu 1}\mu\mu') I_\nu(t_\nu,\mu')\, d\mu' + (1 - c_{\nu 0}) B_\nu(t_\nu) . \tag{5}$$

Here $c_{\nu 0}$ and $c_{\nu 1}$ are defined by

$$c_{\nu 0}=\frac{c_{\nu 1}}{\varpi_1}=\frac{\sigma_\nu(1-\epsilon_\nu)}{\kappa_\nu+\sigma_\nu}. \tag{6}$$

The fact that $B_\nu(t_\nu)$ depends on the radiation intensity leads one, quite naturally, to seek a way in which to remove this non-linearity. One such approach is to assume that $B_\nu(t_\nu)$ can be represented by a linear function of the optical thickness in the continuum (Strömgren 1937; Chandrasekhar 1947, 1950). If, in addition, we assume that σ_ν/κ_ν is independent of depth, then the Planck function can be written as

$$B_\nu(t_\nu) = a_\nu + b_\nu t_\nu\,, \tag{7}$$

where a_ν and b_ν are parameters. The justification of this approximation is discussed by Strömgren (1937).

With equation (7), the transfer equation becomes

$$\mu\frac{\partial}{\partial t}I(t,\mu)+I(t,\mu)=\tfrac{1}{2}\int_{-1}^{1}(c_0+c_1\mu\mu')I(t,\mu')d\mu'+(1-c_0)(a+bt), \tag{8}$$

where, for convenience, the explicit frequency designations have been suppressed. Equation (8) will be solved using the singular eigenfunction expansion technique developed by Case (1960) (see also Case and Zweifel 1967).

Chandrasekhar (1947, 1950) has solved equation (8) for $I(0,-\mu)$, the outgoing angular distribution at the surface of a semi-infinite medium, in the case of isotropic scattering. The purpose of this paper is to include linearly anisotropic scattering effects while deriving both the outgoing distribution at the surface of the semi-infinite medium and the solution $I(t,\mu)$ at any optical distance into the medium.

II. SOLUTION

The solution of equation (8) is developed subject to the two boundary conditions:

a) $I(0,\mu) = 0, \quad \mu \geq 0$ (zero re-entrant radiation),

b) $\lim_{t\to\infty} I(t,\mu) = I_p(t,\mu)\,.$

Here $I_p(t,\mu)$ is the particular solution to the inhomogeneous equation. The particular solution is obtained immediately by assuming it to be a linear function of t. Thus

$$I_p(t,\mu) = a + bt - Q\mu\,, \tag{9}$$

where Q is defined as

$$Q = 3b/(3 - c_1)\,. \tag{10}$$

Using the solutions of the homogeneous equation that were found by Mika (1961), we write the complete solution to equation (8) which satisfies condition (*b*) as

$$I(t,\mu)=A(\eta_0)\phi(\eta_0,\mu)e^{-t/\eta_0}+\int_0^1 A(\eta)\phi(\eta,\mu)e^{-t/\eta}d\eta+a+bt-Q\mu. \tag{11}$$

The eigenmodes are defined by

$$\phi(\eta_0,\mu)=\frac{\eta_0 g(\eta_0,\mu)}{2(\eta_0-\mu)}, \tag{12}$$

$$\phi(\eta,\mu)=\frac{\eta g(\eta,\mu)}{2}\frac{P}{\eta-\mu}+\lambda(\eta)\delta(\eta-\mu) \qquad (-1<\eta<1), \tag{13}$$

and are normalized such that

$$\int_{-1}^{1} \phi(\xi,\mu)\, d\mu = 1 . \tag{14}$$

(In eq. [14] and henceforth, the symbol ξ will denote either η_0 or η, $-1 < \eta < 1$.) Here $A(\eta_0)$ and $A(\eta)$ are the expansion coefficients, while $g(\xi,\mu)$ and $\lambda(\eta)$ are defined by (McCormick and Kuščer 1965)

$$g(\xi,\mu) = c_0 + c_1(1 - c_0)\xi\mu \tag{15}$$

and

$$\lambda(\eta) = g(\eta,\eta)[1 - \eta \tanh^{-1}(\eta)] + (1 - c_0) \qquad (-1 < \eta < 1). \tag{16}$$

The quantity η_0 is the positive root of the equation[1]

$$g(\eta_0,\eta_0)[1 - \eta_0 \tanh^{-1}(1/\eta_0)] + (1 - c_0) = 0. \tag{17}$$

The symbol P denotes that the Cauchy principal value is to be taken when integrals over η or μ are performed. The expansion coefficients must be determined such that the first boundary condition is satisfied. Setting $t = 0$ in equation (11) and applying (a), it follows that

$$Q\mu - a = A(\eta_0)\phi(\eta_0,\mu) + \int_0^1 A(\eta)\phi(\eta,\mu)\, d\eta , \qquad \mu \geq 0 . \tag{18}$$

This is a half-range expansion of the function $(Q\mu - a)$ in terms of $\phi(\eta_0,\mu)$, and $\phi(\eta,\mu)$, $0 < \eta < 1$. Mika (1961) has shown that these eigenfunctions form a complete basis set for functions defined on the half-range $0 \leq \mu \leq 1$. In addition, McCormick and Kuščer (1965, 1966) have shown that the basis set is biorthogonal with respect to the weight function $\mu H(\mu)$, i.e.,

$$\int_0^1 \phi(\xi,\mu)\tilde{\phi}(\xi',\mu)\mu H(\mu)\, d\mu = 0 , \qquad \xi \neq \xi' , \tag{19}$$

where

$$\tilde{\phi}(\xi,\mu) = \phi(\xi,\mu) + \tfrac{1}{2}K\xi \qquad \text{and} \qquad K = \frac{c_1(1 - c_0)c_0\alpha_1}{2 - c_0\alpha_0} . \tag{20}$$

In equation (19), $H(\mu)$ is the H-function of Chandrasekhar (1950). It satisfies an integral equation in the complex z-plane of the form

$$\frac{1}{H(z)} = 1 - \frac{z}{2} \int_0^1 \frac{g(\mu,\mu)H(\mu)\, d\mu}{z + \mu} . \tag{21}$$

Note that $H^{-1}(-\eta_0) = 0$. The α_i in the second of equations (20) denotes moments of the H-function,

$$\alpha_i = \int_0^1 \mu^i H(\mu)\, d\mu , \qquad i = 0,\ 1 . \tag{22}$$

In addition to equation (19), we have the normalization integrals (McCormick and Kuščer 1965, 1966)

$$\int_0^1 \phi(\eta_0,\mu)\tilde{\phi}(\eta_0,\mu)\mu H(\mu)\, d\mu = N(\eta_0) , \tag{23}$$

and

$$\int_0^1 \phi(\eta,\mu)\tilde{\phi}(\eta',\mu)\mu H(\mu)\, d\mu = N(\eta)\delta(\eta - \eta') \quad \text{for} \quad 0 < \eta < 1,\ 0 < \eta' < 1 , \tag{24}$$

[1] Note that η_0 is k^{-1} in the notation of Chandrasekhar (1950).

where

$$N(\eta_0) = \tfrac{1}{2}\eta_0^2 g(\eta_0,\eta_0) H(\eta_0)\left[\frac{g(\eta_0,\eta_0)}{\eta_0(\eta_0^2-1)} - \frac{(1-c_0)g(3\eta_0,\eta_0)}{\eta_0 g(\eta_0,\eta_0)}\right] \tag{25}$$

and

$$N(\eta) = \eta H(\eta)[\lambda^2(\eta) + \tfrac{1}{4}(\pi\eta g(\eta,\eta))^2]. \tag{26}$$

The form of the continuum normalization integral, equation (24), is defined such that orders of integration may be formally reversed, i.e.,

$$\int_0^1 \tilde{\phi}(\eta,\mu)\mu H(\mu)d\mu \int_0^1 A(\eta')\phi(\eta',\mu)d\eta' = A(\eta)N(\eta). \tag{27}$$

With the biorthogonality relations given by equation (19) and the normalization integrals, equations (23) and (24), the expansion coefficients are found immediately by taking scalar products of equation (18). Thus, $A(\eta_0)$ and $A(\eta)$, $0 < \eta < 1$, may be written in the form

$$A(\xi) = \frac{1}{N(\xi)}\int_0^1 (Q\mu - a)\tilde{\phi}(\xi,\mu)\mu H(\mu)d\mu. \tag{28}$$

The above integrals are evaluated by following the approach of McCormick and Kuščer (1966). It is found that

$$\int_0^1 (Q\mu - a)\tilde{\phi}(\xi,\mu)\mu H(\mu)d\mu = -\frac{\xi(Q\xi - a)\Theta(\xi)}{H(-\xi)} + T(\xi), \tag{29}$$

where Θ and T are defined by

$$\begin{aligned}\Theta(\xi) &= 0 \text{ for } 0 < \xi < 1\\ &= 1 \text{ otherwise},\end{aligned} \tag{30}$$

and

$$T(\xi) = Q\xi[(1-c_0)(1-\tfrac{1}{3}c_1)]^{1/2}\left[\xi - \frac{c_0\alpha_1}{2-c_0\alpha_0} - \frac{2a}{Q(2-c_0\alpha_0)}\left(\frac{1-c_0}{1-c_1/3}\right)^{1/2}\right]. \tag{31}$$

Therefore,

$$A(\xi) = \frac{T(\xi)}{N(\xi)}, \tag{32}$$

and the complete solution for the intensity anywhere in the interior of the atmosphere is given by equations (11) and (32):

$$I(t,\mu) = \frac{T(\eta_0)}{N(\eta_0)}\phi(\eta_0,\mu)e^{-t/\eta_0} + \int_0^1 \frac{T(\eta)}{N(\eta)}\phi(\eta,\mu)e^{-t/\eta}d\eta + a + bt - Q\mu. \tag{33}$$

From equation (33), the average intensity and the net flux, defined by

$$J(t) = \tfrac{1}{2}\int_{-1}^1 I(t,\mu)d\mu \quad \text{and} \quad F(t) = 2\int_{-1}^1 I(t,\mu)\mu d\mu, \tag{34}$$

may be found after utilizing equation (14). The results are

$$J(t) = \tfrac{1}{2}\frac{T(\eta_0)}{N(\eta_0)}e^{-t/\eta_0} + \tfrac{1}{2}\int_0^1 \frac{T(\eta)}{N(\eta)}e^{-t/\eta}d\eta + a + bt \tag{35}$$

and

$$F(t) = 2(1-c_0)\left[\eta_0\frac{T(\eta_0)}{N(\eta_0)}e^{-t/\eta_0} + \int_0^1 \eta\frac{T(\eta)}{N(\eta)}e^{-t/\eta}d\eta\right] - \frac{4Q}{3}. \tag{36}$$

The outgoing intensity at the free surface of the atmosphere can be obtained without the necessity of determining explicitly the expansion coefficients, $A(\eta_0)$ and $A(\eta)$; this is done by utilizing a result given by McCormick and Kuščer (1966, eq. [66]), i.e.,

$$I(0,-\mu)=Q\mu+a+\frac{H(\mu)}{\mu}\int_0^1(Q\mu'-a)\tilde{\phi}(-\mu,\mu')\mu' H(\mu')d\mu', \qquad \mu\geq 0 . \tag{37}$$

Equation (29) may be used to evaluate the above integral; it follows that

$$I(0,-\mu)=\frac{H(\mu)}{\mu}T(-\mu), \qquad \mu\geq 0 . \tag{38}$$

For isotropic scattering, this expression reduces to

$$I(0,-\mu)=b(1-c_0)^{1/2}H(\mu)\left[\mu+\frac{c_0 a_1}{2(1-c_0)^{1/2}}+\frac{a}{b}\right], \qquad \mu\geq 0 , \tag{39}$$

which is the result given by Chandrasekhar (1950, § 84, eq. [66]).

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