An exact analytical solution of $x \coth x = \alpha x^2 + 1$

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ABSTRACT

The theory of complex variables is used to develop exact closed-form solutions of the transcendental equation $x \coth x = \alpha x^2 + 1$. *The parameter* α *is considered to be real, and the reported analysis yields analytical expressions, in terms of elementary quadratures, for the real solutions* x, *as they depend on prescribed values of* α.

1. INTRODUCTION

In several recent papers [1-4] we have made use of the theory of complex variables to solve a class of transcendental equations of interest in several areas of mathematical physics. Here we wish to consider the equation

$$x \coth x = \alpha x^2 + 1, \ \alpha > 0, \tag{1}$$

where the parameter $\alpha > 0$ is real and is considered given. As discussed, for example, by Bitter [5] or Smart [6], equation (1) is basic to the Langevin-Weiss theory of ferromagnetism. With this application in mind we therefore seek only the real solutions of equation (1).

If we let

$$x = \frac{2}{a^2}\left[z + \sqrt{z^2 - a^2}\right], \quad a = 2\sqrt{\alpha} \tag{2}$$

where we take the square-root function to be the principal branch in the plane cut from $-a$ to a, then we can, on using some elementary identities, rewrite equation (1) as

$$z + \sqrt{z^2 - a^2} = \frac{a^2}{4} \log\left(\frac{z+1}{z-1}\right). \tag{3}$$

Since the log function in equation (3) is multivalued, it is clear that a solution of equation (3) corresponding to any branch of the log function will yield, by way of equation (2), a solution to equation (1). It can be shown straightforwardly, however, that the desired *real* solutions of equation (1) correspond to the roots of equation (3) with the log function interpreted as the principal branch. We thus seek to establish analytical expressions for the zeros z_k of

$$\lambda(z) = z + \sqrt{z^2 - a^2} - \frac{a^2}{4} \log\left(\frac{z+1}{z-1}\right). \tag{4}$$

2. FACTORIZATION EQUATIONS

It is clear from equation (4) that $\lambda(z)$ is a sectionally analytic function; in fact, for $a < 1$, $\lambda(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis, whereas for $a > 1$, the cut is from $-a$ to a along the real axis. We thus find it convenient to consider separately the two cases (1) $a < 1$ and (2) $a > 1$. First of all, for case (1) we deduce that the boundary values of $\lambda(z)$ as z approaches the cut $[-1, 1]$ from the upper half plane (+) and the lower half plane (-) are given by

$$\lambda^{\pm}(t) = t - \frac{a^2}{4} \ln\left(\frac{1+t}{1-t}\right) \pm i\left[\sqrt{a^2 - t^2} + \frac{\pi a^2}{4}\right],$$
$$t \in (-a, a), \quad a < 1, \tag{5a}$$

and

$$\lambda^{\pm}(t) = t + \operatorname{sgn}(t)\sqrt{t^2 - a^2} - \frac{a^2}{4} \ln\left(\frac{1+t}{1-t}\right) \pm \frac{i\pi a^2}{4},$$
$$t \in (-1, -a) \cup (a, 1), \quad a < 1. \tag{5b}$$

It is clear from equations (5) that the boundary values $\lambda^{\pm}(t)$ cannot vanish on the cut, and thus a straightforward application of the argument principle [7] can now be used to show that $\lambda(z)$ has only two zeros, for $a < 1$, in the finite plane. It is also evident that these zeros occur as real $\pm$ pairs, say $\pm z_0$.

If we now introduce

$$F(z) = \frac{\lambda(z)}{z^2 - z_0^2}, \quad a < 1, \tag{6}$$

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then we conclude that $F(z)$ is analytic in the plane cut from -1 to 1 along the real axis and nonvanishing in the finite plane. Further, on letting z approach the cut from above and below, we deduce that the boundary values of $F(z)$ must satisfy

$$F^+(t) = G(t)\, F^-(t), \quad t \in (-1, 1), \quad a < 1, \tag{7}$$

where

$$G(t) = \frac{\lambda^+(t)}{\lambda^-(t)} = \exp\,[2i\, arg\, \lambda^+(t)]. \tag{8}$$

It therefore follows that $F(z)$ is a canonical solution of the Riemann problem [8] defined by equation (7). The Riemann problem can now be solved in the manner discussed by Muskhelishvili [8], and thus we can write the (properly normalized) canonical solution as

$$F(z) = \frac{2}{z-1} \exp\,[\frac{1}{\pi} \int_{-1}^{1} arg\, \lambda^+(t) \frac{dt}{t-z}], \quad a < 1, \tag{9}$$

where $arg\, \lambda^+(t)$ is continuous for $t \in (-1, 1)$ and defined such that $arg\, \lambda^+(-1) = 0$. Equation (9) can now be entered into equation (6) to yield the factorization

$$z^2 - z_0^2 = \frac{1}{2} \lambda(z)\,(z-1) \exp[-\frac{1}{\pi} \int_{-1}^{1} arg\, \lambda^+(t) \frac{dt}{t-z}], \quad a < 1. \tag{10}$$

We now wish to consider case (2), $a > 1$. Here we find that the boundary values of $\lambda(z)$ take the forms

$$\lambda^{\pm}(t) = t - \frac{a^2}{4} ln\,(\frac{1+t}{1-t}) \pm i[\sqrt{a^2 - t^2} + \frac{\pi a^2}{4}], \quad t \in (-1, 1), \quad a > 1, \tag{11a}$$

and

$$\lambda^{\pm}(t) = t - \frac{a^2}{4} ln\,(\frac{t+1}{t-1}) \pm i\sqrt{a^2 - t^2}, \quad t \in (-a, -1) \cup (a, 1), \quad a > 1. \tag{11b}$$

On using the argument principle again, we find that case (2) subdivides into two cases :

(2a) $a > 1$, $\frac{1}{4} a\, ln\,(\frac{a+1}{a-1}) > 1 \Rightarrow \lambda(z)$ has two real zeros, say $\pm z_0$,

(2b) $a > 1$, $\frac{1}{4} a\, ln(\frac{a+1}{a-1}) < 1 \Rightarrow \lambda(z)$ has no zeros.

It is also evident here that $\lambda^{\pm}(t)$ cannot vanish for $t \in (-a, a)$. Of course, case (2b) is of no interest since equation (1) has no real solutions. On the other hand, for case (2a) the function

$$F(z) = \frac{\lambda(z)}{z^2 - z_0^2}, \quad a > 1 \text{ and } \frac{1}{4} a\, ln\,(\frac{a-1}{a+1}) > 1, \tag{12}$$

must be a canonical solution of the Riemann problem

$$F^+(t) = G(t)\, F^-(t), \; t \in (-a, a), \quad a > 1 \text{ and } \frac{1}{4} a\, ln\,(\frac{a-1}{a+1}) > 1, \tag{13}$$

where

$$G(t) = \frac{\lambda^+(t)}{\lambda^-(t)} = \exp\,[2i\, arg\, \lambda^+(t)] \tag{14}$$

If now we take $arg\, \lambda^+(t)$ to be continuous, such that $arg\, \lambda^+(-a) = 0$, we can write

$$F(z) = \frac{2}{z-a} \exp\,[\frac{1}{\pi} \int_{-a}^{a} arg\, \lambda^+(t) \frac{dt}{t-z}], \quad a > 1 \text{ and } \frac{1}{4} a\, ln\,(\frac{a-1}{a+1}) > 1, \tag{15}$$

and subsequently we deduce the factorization

$$z^2 - z_0^2 = \frac{1}{2} \lambda(z)(z-a) \exp[-\frac{1}{\pi} \int_{-a}^{a} arg\, \lambda^+(t) \frac{dt}{t-z}], \quad a > 1 \text{ and } \frac{1}{4} a\, ln\,(\frac{a-1}{a+1}) > 1. \tag{16}$$

3. EXPLICIT SOLUTIONS

Having established the factorization equation, we can now enter equation (10) or equation (16) into equation (2) to obtain the explicit solution of equation (1). First of all, for case 1, $\alpha < 1/4$, we can evaluate equation (10) at say $z = \xi$ and enter that result into equation (2) to obtain the exact solution

$$x_0 = \pm \frac{1}{2\alpha} [\sqrt{\xi^2 - K(\xi)} + \sqrt{\xi^2 - 4\alpha - K(\xi)}], \quad \alpha < \frac{1}{4}, \tag{17}$$

where

$$K(\xi) = \frac{1}{2}(\xi - 1)[\xi + \sqrt{\xi^2 - 4\alpha} - \alpha \log(\frac{\xi+1}{\xi-1})] \times \exp[-\frac{1}{\pi} \int_{-1}^{1} arg\, \lambda^+(t) \frac{dt}{t-\xi}] \tag{18}$$

and $\log z$ denotes the principal branch of the log function. We note that equation (17) is an identity in ξ, and thus the equation yields an analytical solution to equation (1) for any convenient value of ξ. We list here the two special forms resulting from setting $\xi = 0$ and $\xi = \infty$ in equation (17) :

$$x_0 = \pm \frac{1}{2\alpha} [\sqrt{L} + \sqrt{L - 4\alpha}], \quad \alpha < \frac{1}{4}, \tag{19}$$

where

$$L = (\alpha \frac{\pi}{2} + \sqrt{\alpha}) \exp\,[-\frac{1}{\pi} \int_{-1}^{1} [arg\, \lambda^+(t) - \frac{\pi}{2}] \frac{dt}{t}], \tag{20}$$

and

$$x_0 = \pm \frac{1}{2\alpha}[\sqrt{\frac{1}{2} + 2\alpha - M} + \sqrt{\frac{1}{2} - 2\alpha - M}],$$
$$\alpha < \frac{1}{4}, \qquad (21)$$

where

$$M = \frac{1}{\pi}\int_{-1}^{1} t \, arg \, \lambda^+(t) \, dt. \qquad (22)$$

Considering now that $\alpha > 1/4$, we note that equation (1) has a real solution only if case (2a) is applicable. For this case we can evaluate equation (16) at $z = \xi$ and enter that result into equation (1) to obtain

$$x_0 = \pm \frac{1}{2\alpha}[\sqrt{\xi^2 - P(\xi)} + \sqrt{\xi^2 - 4\alpha - P(\xi)}],$$
$$\alpha > \frac{1}{4} \text{ and } \coth \frac{1}{\sqrt{\alpha}} > 2\sqrt{\alpha}, \qquad (23)$$

where

$$P(\xi) = \frac{1}{2}(\xi - 2\sqrt{\alpha})\,[\xi + \sqrt{\xi^2 - 4\alpha}$$
$$-\alpha \log(\frac{\xi+1}{\xi-1})] \exp[-\frac{1}{\pi}\int_{-2\sqrt{\alpha}}^{2\sqrt{\alpha}} arg \, \lambda^+(t) \frac{dt}{t-\xi}]. \qquad (24)$$

Here equation (23) is a solution to equation (1) for any convenient value of ξ; we again list the special forms due to setting $\xi = 0$ and $\xi = \infty$:

$$x_0 = \pm \frac{1}{2\alpha}[\sqrt{Q} + \sqrt{Q - 4\alpha}],$$
$$\alpha > \frac{1}{4} \text{ and } \coth \frac{1}{\sqrt{\alpha}} > 2\sqrt{\alpha}, \qquad (25)$$

where

$$Q = \alpha(2 + \pi\sqrt{\alpha}) \exp[-\frac{1}{\pi}\int_{-2\sqrt{\alpha}}^{2\sqrt{\alpha}} [arg \, \lambda^+(t) - \frac{\pi}{2}]\frac{dt}{t}], \qquad (26)$$

and

$$x_0 = \pm \frac{1}{2\alpha}[\sqrt{4\alpha - R} + \sqrt{-R}],$$
$$\alpha > \frac{1}{4} \text{ and } \coth \frac{1}{\sqrt{\alpha}} > 2\sqrt{\alpha}, \qquad (27)$$

where

$$R = \frac{1}{\pi}\int_{-2\sqrt{\alpha}}^{2\sqrt{\alpha}} t \, arg \, \lambda^+(t) \, dt. \qquad (28)$$

In conclusion we note that the general equations (17) and (23) are our explicit closed-form solutions to equation (1); whereas equation (19) and (21), and (23) and (25), are the special forms obtained from special choices of the free parameter ξ. A Gaussian quadrature scheme has been used to evaluate all of our explicit solutions, for selected values of α, and quite straightforwardly solutions accurate to six significant figures were obtained.

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