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Neutron Transport in Two Dissimilar Media with Anisotropic Scattering

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The elementary solutions of the one-speed neutron-transport equation with linearly anisotropic scattering are used in conjunction with Chandrasekhar's invariance principles to solve in a concise manner the Milne problem for two adjoining half-spaces and the critical reactor problem for a reflected slab.

I. INTRODUCTION

The elementary solutions of Case[1] were used by Kuszell[2] to study neutron diffusion for problems defined by the presence of two dissimilar media. That work, however, was limited to isotropic scattering and was left in a somewhat cumbersome final form. Later work by Mendelson and Summerfield[3] added to the general area of multiregion problems; however, it is to the basic work of McCormick[4] and McCormick and Doyas[5] that we must look for the most significant contribution to two-media problems with the effects of anisotropic scattering included. Today in the field of neutron-transport theory many researchers consider the fundamental paper by Case[1] to be the cornerstone of the theory of "exact" solutions. Later Pahor and Zweifel[6] in an elegant paper demonstrated how the work of Chandrasekhar[7] and Case[1] could be coupled and utilized at the same time to obtain in a profitable and concise manner certain results for a variety of single-medium problems.

In a recent Note, Siewert and Burkart[8] demonstrated how the principles of invariance, as developed by Chandrasekhar,[7] could be used effectively to analyze the critical reactor problem for a reflected slab with isotropic scattering. In this paper, we wish to show explicitly the complications that arise when the same critical problem and the Milne problem for two adjoining half-spaces are solved for the case of linearly anisotropic scattering.

II. THE MILNE PROBLEM FOR TWO HALF-SPACES

We consider the one-speed neutron-transport equation for region 1, $x > 0$, and region 2, $x < 0$, written in the familar manner

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[1]K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).
[2]A. KUSZELL, *Acta Phys. Pol.*, **20**, 567 (1961).
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[4]N. J. McCORMICK, *Nucl. Sci. Eng.*, **37**, 243 (1969).
[5]N. J. McCORMICK and R. J. DOYAS, *Nucl. Sci. Eng.*, **37**, 252 (1969).
[6]S. PAHOR and P. F. ZWEIFEL, *J. Math. Phys.*, **10**, 581 (1969).
[7]S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, London (1950).
[8]C. E. SIEWERT and A. R. BURKART, *Nucl. Sci. Eng.*, **58**, 253 (1975).

$$\mu \frac{\partial}{\partial x} \Psi_\alpha(x,\mu) + \Psi_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} \Psi_\alpha(x,\mu')(1 + b_\alpha \mu\mu')\,d\mu' \quad . \tag{1}$$

Here $\Psi_\alpha(x,\mu)$ denotes the neutron angular density in region α, as a function of position (in optical units) x and the direction cosine of the propagating neutrons, μ. In addition, c_α denotes the mean number of secondary neutrons per collision in region α, and b_α is the coefficient of anisotropic scattering.

For the considered Milne problem, we seek a diverging (as $x \to \infty$) solution of Eq. (1) such that

$$\lim_{x\to\infty} \Psi_1(x,\mu)\exp(-x/\nu_0) < \infty \quad , \tag{2a}$$

$$\lim_{x\to-\infty} \Psi_2(x,\mu) = 0 \tag{2b}$$

and

$$\Psi_1(0,\mu) = \Psi_2(0,\mu) \quad , \quad \mu \in (-1,1) \quad . \tag{2c}$$

Here ν_0 denotes the discrete eigenvalue in region 1. (We use, with only slight modification, the notation of Case and Zweifel,[9] so that many of the basic quantities need not be redefined here.)

Relying on the basic work of McCormick and Kuščer,[10] we can immediately write solutions to Eq. (1) that satisfy the boundary conditions listed as Eqs. (2a) and (2b):

$$\Psi_1(x,\mu) = A(\nu_0)\,\phi_1(\nu_0,\mu)\exp(-x/\nu_0) + \phi_1(-\nu_0,\mu)\exp(x/\nu_0) + \int_0^1 A(\nu)\,\phi_1(\nu,\mu)\exp(-x/\nu)\,d\nu \tag{3a}$$

and

$$\Psi_2(x,\mu) = B(-\eta_0)\,\phi_2(-\eta_0,\mu)\exp(x/\eta_0) + \int_0^1 B(-\eta)\,\phi_2(-\eta,\mu)\exp(x/\eta)\,d\eta \quad . \tag{3b}$$

Here

$$\phi_\alpha(\xi_\alpha,\mu) = \frac{c_\alpha \xi_\alpha}{2} R_\alpha(\xi_\alpha,\mu)\frac{1}{\xi_\alpha - \mu} \quad , \tag{4}$$

where

$$R_\alpha(x,y) = 1 + l_\alpha xy$$
$$l_\alpha = b_\alpha(1 - c_\alpha)$$
$$\xi_1 = \nu_0$$
$$\xi_2 = \eta_0 \quad ,$$

[9]K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Company, Reading, Massachusetts (1967).

[10]N. J. McCORMICK and I. KUŠČER, *J. Math. Phys.*, **6**, 1939 (1965).

and where $\xi_\alpha \notin (-1,1)$ is the positive zero of

$$\Lambda_\alpha(z) = 1 - c_\alpha z R_\alpha(z,z)\tanh^{-1}\frac{1}{z} + c_\alpha l_\alpha z^2 \quad . \tag{5}$$

Also,

$$\phi_\alpha(\xi,\mu) = \frac{c_\alpha \xi}{2} R_\alpha(\xi,\mu)\frac{P}{\xi - \mu} + \lambda_\alpha(\xi)\,\delta(\xi - \mu) \quad , \quad \xi \in (-1,1) \quad , \tag{6}$$

with

$$\lambda_\alpha(\xi) = 1 - c_\alpha \xi R_\alpha(\xi,\xi)\tanh^{-1}\xi + c_\alpha l_\alpha \xi^2 \quad . \tag{7}$$

Since the solutions given by Eqs. (3) inherently satisfy Eqs. (2a) and (2b), we need simply to constrain them to obey Eq. (2c), which we choose to write as

$$\Psi_1(0,\mu) = \Psi_2(0,\mu) \quad \text{and} \quad \Psi_1(0,-\mu) = \Psi_2(0,-\mu) \quad , \quad \mu \in (0,1) \quad . \tag{8}$$

At this point we can use the S function of Chandrasekhar[7] to write

$$\Psi_2(0,\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\,\Psi_2(0,-\mu')\,d\mu' \quad , \quad \mu \in (0,1) \quad , \tag{9}$$

where

$$S_2(\mu',\mu) = \frac{c_2\mu\mu'}{\mu + \mu'}[1 - \hat{c}_2(\mu + \mu') - l_2\mu\mu'] \times H_2(\mu')\,H_2(\mu) \quad . \tag{10}$$

Here $H_2(\mu)$ is Chandrasekhar's H function for region 2 and, in general,

$$\hat{c}_\alpha = \frac{c_\alpha l_\alpha \alpha_{\alpha,1}}{2 - c_\alpha \alpha_{\alpha,0}} \quad ,$$

$$\hat{q}_\alpha = \frac{2(1 - c_\alpha)}{2 - c_\alpha \alpha_{\alpha,0}} \quad ,$$

and

$$\alpha_{\alpha,\beta} = \int_0^1 H_\alpha(\mu)\,\mu^\beta\,d\mu \quad . \tag{11}$$

If we now enter Eq. (8) into Eq. (9), we can obtain

$$\Psi_1(0,\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\,\Psi_1(0,-\mu')\,d\mu' \quad , \quad \mu \in (0,1) \quad . \tag{12}$$

We consider that Eq. (12) is the basic equation now to be satisfied, since if $A(\nu_0)$ and $A(\nu)$ are established by Eq. (12), then $B(-\eta_0)$ and $B(-\eta)$ can be obtained immediately from Eq. (8) by using the half-range orthogonality relations of McCormick and Kuščer.[10]

On substituting Eq. (3a) into Eq. (12), we find that we can evaluate the integral over μ' to obtain

$$\frac{A(\nu_0)}{H_2(\nu_0)}\left[\phi_1(\nu_0,\mu) - W(\nu_0)\right] + \int_0^1 \frac{A(\nu)}{H_2(\nu)}\left[\phi_1(\nu,\mu) - W(\nu)\right]d\nu = -\frac{1}{H_2(-\nu_0)}\left[\phi_1(-\nu_0,\mu) - W(-\nu_0)\right] \quad , \quad \mu \in (0,1) \quad , \tag{13}$$

where

$$W(\xi) = \frac{1}{2}\frac{c_1\xi}{R_2(\xi,\xi)}\{\xi(l_1 - l_2)[\hat{q}_2H_2(\xi) - 1] - \hat{c}_2R_1(\xi,\xi)\} \quad . \tag{14}$$

Equation (13) clearly is a singular integral equation that can be regularized by using the half-range orthogonality relations for one medium (McCormick and Kuščer[10]). Thus, if we multiply Eq. (13) by

$$\left[\phi_1(\nu_0,\mu) + \frac{c_1\nu_0}{2}\hat{c}_1\right]\mu H_1(\mu)$$

and integrate over μ, we find

$$\frac{A(\nu_0)}{H_2(\nu_0)}\left[N_1(\nu_0)H_1(\nu_0) - \nu_0\hat{q}_1W(\nu_0)\right] - \nu_0\hat{q}_1\overline{A} = -\frac{1}{H_2(-\nu_0)}\left[J(-\nu_0,\nu_0) - \nu_0\hat{q}_1W(-\nu_0)\right] \quad , \tag{15}$$

where

$$\overline{A} = \int_0^1 \frac{A(\nu)}{H_2(\nu)}W(\nu)\,d\nu \quad , \tag{16}$$

$$N_1(\nu_0) = \frac{c_1\nu_0^2}{2}R_1(\nu_0,\nu_0)\left[\frac{c_1R_1(\nu_0,\nu_0)}{\nu_0(\nu_0^2 - 1)} - \frac{(1 - c_1)R_1(3\nu_0,\nu_0)}{\nu_0R_1(\nu_0,\nu_0)}\right] \quad , \tag{17}$$

and

$$J(-\nu_0,\xi) = \frac{c_1\nu_0\xi}{2(\nu_0 + \xi)H_1(\nu_0)} \times \left[1 - l_1\nu_0\xi + \hat{c}_1(\nu_0 + \xi)\right] \quad . \tag{18}$$

In a similar manner, we can multiply Eq. (13) by

$$\left[\phi_1(\nu',\mu) + \frac{c_1\nu'}{2}\hat{c}_1\right]\mu H_1(\mu) \quad , \quad \nu' \in (0,1) \quad ,$$

and integrate to obtain

$$-\frac{A(\nu_0)}{H_2(\nu_0)}\nu'\hat{q}_1W(\nu_0) + \frac{A(\nu')}{H_2(\nu')}N_1(\nu')H_1(\nu') - \nu'\hat{q}_1\overline{A} = -\frac{1}{H_2(-\nu_0)}\left[J(-\nu_0,\nu') - \nu'\hat{q}_1W(-\nu_0)\right] \quad . \tag{19}$$

Here,

$$N_1(\nu) = \nu\left\{[\lambda_1(\nu)]^2 + \left[\frac{c_1\nu\pi}{2}R_1(\nu,\nu)\right]^2\right\} \quad . \tag{20}$$

If now we rearrange Eq. (19) and multiply by $W(\nu')$, we can integrate to find

$$\overline{A} = -\frac{c_1K_1R_1(\nu_0,\nu_0)}{4H_1(\nu_0)H_2(-\nu_0)} + \frac{A(\nu_0)H_1(\nu_0)N_1(\nu_0)K_2}{\nu_0H_2(\nu_0)} \quad , \tag{21}$$

where the two constants K_1 and K_2 are given by

$$K_1 = \int_0^1 \frac{\nu W(\nu)(\nu_0 - \nu)}{N_1(\nu)H_1(\nu)(\nu_0 + \nu)}\,d\nu \tag{22a}$$

and

$$K_2 = \int_0^1 \frac{\nu W(\nu)}{N_1(\nu)H_1(\nu)}\,d\nu \quad . \tag{22b}$$

Equation (21) can be used in Eq. (15) to find $A(\nu_0)$, and subsequently $A(\nu)$ can be found from Eq. (19):

$$A(\nu_0) = -\frac{H_2(\nu_0)}{H_2(-\nu_0)} \times \frac{\{c_1\nu_0[1 - l_1\nu_0^2 + 2\nu_0\hat{c}_1 + \hat{q}_1K_1R_1(\nu_0,\nu_0)] - 4\nu_0H_1(\nu_0)\hat{q}_1W(-\nu_0)\}}{[4N_1(\nu_0)H_1(\nu_0)H_1(\nu_0)(1 - \hat{q}_1K_2) - 4\nu_0H_1(\nu_0)\hat{q}_1W(\nu_0)]} \tag{23}$$

and

$$A(\nu) = \frac{\nu H_2(\nu)}{N_1(\nu)H_1(\nu)}\left[\frac{A(\nu_0)H_1(\nu_0)N_1(\nu_0)}{\nu_0H_2(\nu_0)} - \frac{c_1(\nu_0 - \nu)R_1(\nu_0,\nu_0)}{4(\nu_0 + \nu)H_2(-\nu_0)H_1(\nu_0)}\right] \quad . \tag{24}$$

Equations (23) and (24) are explicit expressions for the expansion coefficients $A(\nu_0)$ and $A(\nu)$. Though our final results were obtained differently and are different in appearance from those of McCormick[4] and McCormick and Doyas,[5] they are similar in that there appear extra terms, in our case, W terms, in regard to either the isotropic scattering case, $l_1 = l_2 = 0$, or the single medium result, $c_2 = 0$.

We note that the neutron density in region 1 is given by

$$\rho_1(x) = \int_{-1}^1 \Psi_1(x,\mu)\,d\mu = A(\nu_0)\exp(-x/\nu_0) + \exp(x/\nu_0) + \int_0^1 A(\nu)\exp(-x/\nu)\,d\nu \quad . \tag{25}$$

Also, an asymptotic solution can be written as

$$\rho_{1a}(x) = A(\nu_0)\exp(-x/\nu_0) + \exp(x/\nu_0) \quad . \tag{26}$$

Thus, if we write

$$z_0 = -\frac{\nu_0}{2}\ln[-A(\nu_0)] \quad , \tag{27}$$

then Eq. (26) becomes

$$\rho_{1a}(x) = \exp(x/\nu_0) - \exp[-(x + 2z_0)/\nu_0] \quad . \tag{28}$$

It is therefore clear that z_0 is the extrapolated endpoint for the considered Milne problem.

III. THE CRITICAL PROBLEM FOR A REFLECTED SLAB

We consider the transport equations for the core, $-a \leq x \leq a$, and the reflector, $|x| > a$, written as Eq. (1), where $\alpha = 1$ implies the core and $\alpha = 2$ implies the reflector. Clearly we take $c_1 > 1$ and $c_2 < 1$. We thus seek solutions of Eq. (1) such that $\Psi_\alpha(-x, -\mu) = \Psi_\alpha(x, \mu)$, $\Psi_1(a, \mu) = \Psi_2(a, \mu)$, $\mu \in (-1, 1)$, and $\Psi_2(\infty, \mu) = 0$. We consider that c_1 and c_2 are given and thus seek the critical half-thickness a.

For the core, we can write the desired solution as

$$\Psi_1(x, \mu) = \hat{A}(\nu_0)[\phi_1(\nu_0, \mu)\exp(-x/\nu_0) + \phi_1(-\nu_0, \mu)\exp(x/\nu_0)] + \int_0^1 \hat{A}(\nu)[\phi_1(\nu, \mu)\exp(-x/\nu) + \phi_1(-\nu, \mu)\exp(x/\nu)]\,d\nu \quad , \tag{29}$$

which clearly satisfies the symmetry condition. For the reflector, we need only consider $x > a$, and thus we write

$$\Psi_2(x, \mu) = \hat{B}(\eta_0)\,\phi_2(\eta_0, \mu)\exp(-x/\eta_0) + \int_0^1 \hat{B}(\eta)\,\phi_2(\eta, \mu)\exp(-x/\eta)\,d\eta \quad , \quad x > a \quad . \tag{30}$$

As in a previous work for isotropic scattering (Siewert and Burkart[8]), we would now like to use the continuity condition at $x = a$ in the reflection equation,

$$\Psi_2(a, -\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu', \mu)\,\Psi_2(a, \mu')\,d\mu' \quad , \quad \mu \in (0, 1) \quad , \tag{31}$$

to obtain what we consider to be our basic boundary condition:

$$\Psi_1(a, -\mu) = \frac{1}{2\mu}\int_0^1 S_2(\mu', \mu)\,\Psi_1(a, \mu')\,d\mu' \quad , \quad \mu \in (0, 1) \quad . \tag{32}$$

If we now substitute Eq. (29) into Eq. (32), we can evaluate the integral over μ' to obtain

$$\frac{\hat{A}(\nu_0)}{H_2(\nu_0)}\exp(a/\nu_0)[\phi_1(\nu_0, \mu) - W(\nu_0)] + \int_0^1 \frac{\hat{A}(\nu)}{H_2(\nu)}\exp(a/\nu)[\phi_1(\nu, \mu) - W(\nu)]\,d\nu$$
$$= -\frac{\hat{A}(\nu_0)}{H_2(-\nu_0)}\exp(-a/\nu_0)[\phi_1(-\nu_0, \mu) - W(-\nu_0)] - \int_0^1 \hat{A}(\nu)\exp(-a/\nu)[H_2(\nu)K(\nu)\,\phi_1(-\nu, \mu) - G(\nu)]\,d\nu \quad , \quad \mu \in (0, 1) \quad , \tag{33}$$

where

$$K(\nu) = \frac{c_2(c_1 - 1)R_2(\nu, \nu) - c_1(c_2 - 1)R_1(\nu, \nu)}{c_1 R_1(\nu, \nu)} \tag{34}$$

and

$$G(\nu) = \frac{c_1\nu}{2R_2(\nu, \nu)}\{H_2(\nu)K(\nu)[\hat{c}_2 R_1(\nu, \nu) - \nu(l_1 - l_2)] + \nu(l_1 - l_2)\hat{q}_2\} \quad . \tag{35}$$

Equation (33) is a singular integral equation that can be regularized by using the half-range orthogonality relations of McCormick and Kuščer.[10] Thus, we multiply Eq. (33) by

$$\mu H_1(\mu)\left[\phi_1(\nu_0, \mu) + \frac{c_1\nu_0}{2}\hat{c}_1\right]$$

and integrate to obtain

$$\frac{\exp(2a/\nu_0)}{H_2(\nu_0)}[N_1(\nu_0)H_1(\nu_0) - \nu_0\hat{q}_1 W(\nu_0)] - \nu_0\hat{q}_1\overline{D} = -\frac{1}{H_2(-\nu_0)}[J(-\nu_0, \nu_0) - \nu_0\hat{q}_1 W(-\nu_0)] - \int_0^1 D(\nu)\exp(-2a/\nu)$$
$$\times [H_2(\nu)K(\nu)J(-\nu, \nu_0) - \nu_0\hat{q}_1 G(\nu)]\,d\nu \quad , \tag{36}$$

where we have introduced

$$D(\nu) = \frac{\hat{A}(\nu)\exp(a/\nu)\exp(a/\nu_0)}{\hat{A}(\nu_0)} \quad \text{and} \quad \overline{D} = \int_0^1 \frac{D(\nu)}{H_2(\nu)}W(\nu)\,d\nu \quad . \tag{37}$$

In a similar manner, we can multiply Eq. (33) by

$$\mu H_1(\mu)\left[\phi_1(\nu', \mu) + \frac{c_1\nu'}{2}\hat{c}_1\right] \quad , \quad \nu' \in (0, 1) \quad ,$$

and integrate to get

$$-\frac{\exp(2a/\nu_0)}{H_2(\nu_0)}\,\nu'\hat{q}_1 W(\nu_0) + \frac{D(\nu')}{H_2(\nu')}\,N_1(\nu')\,H_1(\nu')$$

$$-\,\nu'\hat{q}_1\overline{D} = -\frac{1}{H_2(-\nu_0)}\left[J(-\nu_0,\nu') - \nu'\hat{q}_1 W(-\nu_0)\right]$$

$$-\int_0^1 D(\nu)\exp(-2a/\nu)\left[H_2(\nu)\,K(\nu)\,J(-\nu,\nu')\right.$$

$$\left.-\,\nu'\hat{q}_1 G(\nu)\right]d\nu \quad , \qquad \nu' \in (0,1) \quad , \qquad (38)$$

where $J(-\nu, \xi)$ follows from Eq. (18) after changing ν_0 to ν. Equations (36) and (38) are the two regular integral equations that we must solve simultaneously to obtain the critical half-thickness a. Though it is perhaps unreasonable to expect to be able to find analytical solutions to Eqs. (36) and (38), to construct a numerical solution certainly poses no problem.

Before listing our final results, obtained by solving Eqs. (36) and (38) iteratively, we note that there are two immediately available approximations we can introduce. The simpler is to set $D(\nu) = 0$ and ignore completely Eq. (38). This approximation leads to a critical condition that yields the approximate critical half-thickness

$$a_{0B} = \tfrac{1}{2}\pi|\nu_0| - z_{0B} \quad , \qquad (39)$$

re

$$z_{0B} = \frac{\nu_0}{2}\log\left\{\frac{H_2(-\nu_0)}{H_2(\nu_0)}\left[4N_1(\nu_0)\,H_1(\nu_0)\,H_1(\nu_0)\right.\right.$$

$$\left.-\,4\nu_0 H_1(\nu_0)\,\hat{q}_1 W(\nu_0)\right] \div \left[c_1\nu_0(1 - l_1\nu_0^2 + 2\nu_0\hat{c}_1)\right.$$

$$\left.\left.-\,4\nu_0 H_1(\nu_0)\,\hat{q}_1 W(-\nu_0)\right]\right\} \quad . \qquad (40)$$

On the other hand, we might approximate Eqs. (36) and (38) in such a way that we can utilize the results for the Milne problem developed in Sec. II. We observe that if we neglect the integral terms on the right-hand sides of Eqs. (36) and (38), then the *two* approximated equations will have the solutions

$$D_M(\nu) = A(\nu)$$

and

$$\exp(2a_{0M}/\nu_0) = A(\nu_0) \quad , \qquad (41)$$

where $A(\nu_0)$ and $A(\nu)$ are the Milne solutions given by Eqs. (23) and (24). From Eq. (41), we get a second approximate half-thickness

$$a_{0M} = \tfrac{1}{2}\pi|\nu_0| - z_{0M} \quad , \qquad (42)$$

where

$$z_{0M} = \frac{\nu_0}{2}\log\left(\frac{H_2(-\nu_0)}{H_2(\nu_0)}\left[4N_1(\nu_0)\,H_1(\nu_0)\,H_1(\nu_0)(1 - \hat{q}_1 K_2)\right.\right.$$

$$\left.-\,4\nu_0 H_1(\nu_0)\hat{q}_1 W(\nu_0)\right] \div \left\{c_1\nu_0\left[1 - l_1\nu_0^2\right.\right.$$

$$\left.\left.\left.+\,2\nu_0\hat{c}_1 + \hat{q}_1 K_1 R_1(\nu_0,\nu_0)\right] - 4\nu_0 H_1(\nu_0)\,\hat{q}_1 W(-\nu_0)\right\}\right) \quad . \qquad (43)$$

For the case of isotropic scattering, $l_1 = l_2 = 0$, or for the case of a bare reactor, $c_2 = 0$, the two approximations given by Eqs. (39) and (42) are identical, but in general they are different. In Table I, we list some typical values of z_{0B} and z_{0M}, and in Table II, we list a_{0B}, a_{0M} and our "exact" results obtained by solving Eqs. (36) and (38) iteratively. We note that our exact results are identical to those reported by Caroll and

TABLE I

Extrapolated Endpoints

c_1	b_1	c_2	b_2	z_{0B}	z_{0M}
1.01	0.0	0.9	0.0	1.818175	1.818175
1.01	1.0	0.9	0.0	2.697328	2.697147
1.01	0.0	0.9	1.0	1.533918	1.537910
1.01	1.0	0.9	1.0	2.285898	2.291668
1.06	0.0	0.9	0.0	1.564092	1.564092
1.06	0.0	0.9	1.0	1.366538	1.370686
1.20	0.0	0.4	0.0	0.6992548	0.6992548
1.20	1.0	0.4	0.0	1.091650	1.090817
1.20	0.0	0.4	1.0	0.660828	0.6631535
1.20	1.0	0.4	1.0	1.037642	1.040783
1.50	0.0	0.9	0.0	0.7914220	0.7914220
1.50	1.0	0.9	0.0	1.190381	1.182627
1.50	0.0	0.9	1.0	0.7479395	0.7516228
1.50	1.0	0.9	1.0	1.142157	1.140871
1.60	0.0	0.4	0.0	0.5097969	0.5097969
1.60	1.0	0.4	0.0	0.8605117	0.8578493
1.60	0.0	0.4	1.0	0.4900796	0.4920241
1.60	1.0	0.4	1.0	0.8343955	0.8360061

TABLE II

Critical Sizes

c_1	b_1	c_2	b_2	a_{0B}	a_{0M}	a
1.01	0.0	0.9	0.0	7.214751	7.214751	7.214751
1.01	1.0	0.9	0.0	8.393351	8.393532	8.393532
1.01	0.0	0.9	1.0	7.499008	7.495017	7.495017
1.01	1.0	0.9	1.0	8.804780	8.799010	8.799010
1.06	0.0	0.9	0.0	2.052395	2.052395	2.052360
1.06	0.0	0.9	1.0	2.249949	2.245801	2.245784
1.20	0.0	0.4	0.0	1.182975	1.182975	1.182419
1.20	1.0	0.4	0.0	1.328226	1.329059	1.328237
1.20	0.0	0.4	1.0	1.221402	1.219077	1.218626
1.20	1.0	0.4	1.0	1.382234	1.379093	1.378425
1.50	0.0	0.9	0.0	0.2910616	0.2910616	0.2825876
1.50	1.0	0.9	0.0	0.3002338	0.3079874	0.2889746
1.50	0.0	0.9	1.0	0.3345442	0.3308609	0.3248086
1.50	1.0	0.9	1.0	0.3484569	0.3497437	0.3347018
1.60	0.0	0.4	0.0	0.4509333	0.4509333	0.4468885
1.60	1.0	0.4	0.0	0.4922023	0.4948646	0.4817374
1.60	0.0	0.4	1.0	0.4706506	0.4687061	0.4653584
1.60	1.0	0.4	1.0	0.5183184	0.5167079	0.5051470

Aronson.[11] Also, z_{0M} and a_{0M} agree with the asymptotic results of Doyas and McCormick.[12] Numerical results in addition to those given here can be found in the thesis of Burkart.[13]

The results listed in Table II indicate that the two approximate values a_{0B} and a_{0M} are correct to three significant figures for the listed values of $c_1 \leq 1.2$. For the larger values of c_1 listed in Table II, a_{0B} and a_{0M} are accurate only to one or two significant figures. The results of Table II also illustrate the fact that neither a_{0B} nor a_{0M} is always the better approximation.

Finally, we note that Erdmann[14] has reported an extensive study concerning the consistency of certain approximations related to critical problems for bare slabs and spheres.

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