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Technical Notes

An Improved P-L Solution to the Reflected Critical-Reactor Problem in Slab Geometry

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ABSTRACT

Principles of invariance are used with the traditional P-L method to yield concise and improved results for critical calculations of reflected reactors.

ANALYSIS

The purpose of this Note is to report the results of a simple approximate solution to the reflected critical-slab problem. The technique blends the features of the Chandrasekhar H function[1] with the usual P-L approximation to give a result for the critical dimension that is consistently more accurate and, in a sense, less complicated than that obtained from the traditional P-L method.

We seek a solution to

$$\mu \frac{\partial}{\partial x} \psi_\alpha(x,\mu) + \psi_\alpha(x,\mu) = \tfrac{1}{2} c_\alpha \int_{-1}^{1} (1 + b_\alpha \mu\mu')\psi_\alpha(x,\mu')\,d\mu' \quad , \tag{1}$$

where $\psi_1(x,\mu)$ denotes the angular flux in the core, $x\epsilon(-a, a)$, and where $\psi_2(x,\mu)$ is the angular flux in the infinite reflector, $|x| > a$. Here we consider that c_1, c_2, b_1, and b_2 are given, and thus we seek the value of a for which there exists a real non-negative solution of Eq. (1), subject to the boundary conditions

$$\psi_\alpha(x,\mu) = \psi_\alpha(-x,-\mu) \quad , \tag{2a}$$

$$\lim_{|x|\to\infty} \psi_2(x,\mu) = 0 \quad , \tag{2b}$$

and

$$\psi_1(a,\mu) = \psi_2(a,\mu) \quad , \quad \mu\epsilon(-1,1) \quad . \tag{2c}$$

We note that a rigorous solution of this problem is reported in a related paper[2]; but here, we wish to give a simple approximate solution that can be useful for low-order calculations. For the core, we can write the properly symmetric P-L solution, for L odd, as

$$\psi_1(x,\mu) = \sum_{l=0}^{L} \sum_{j=1}^{[(L+1)/2]} \left(\frac{2l+1}{2}\right) P_l(\mu) T_l(\nu_j) A_j \times [\exp(-x/\nu_j) + (-1)^l \exp(x/\nu_j)] \quad , \tag{3}$$

where $P_l(\mu)$ denotes the Legendre polynomial, the arbitrary coefficients A_j are to be determined from the boundary conditions, and the polynomials $T_l(\xi)$ follow from[1]

$$[2l + 1 - c_1(\delta_{l,0} + b_1 \delta_{l,1})]\, \xi T_l(\xi) = (l+1) T_{l+1}(\xi) + l T_{l-1}(\xi) \quad , \tag{4}$$

with $T_0(\xi) = 1$. Also, the eigenvalues ν_j required in Eq. (3) are the "positive" $\frac{1}{2}(L+1)$ zeros of $T_{L+1}(\xi)$.

Were we to pursue the traditional P-L approximation, we would now write an expression similar to Eq. (3) for the reflector, equate moments of the two expressions evaluated at $x = a$, and thus obtain a critical condition that the determinant of the resulting $(L+1)\times(L+1)$ coefficient matrix must vanish. Here, however, since $c_2 < 1$, we can use the Chandrasekhar result,[1]

$$\psi_2(a,-\mu) = \frac{1}{2\mu} \int_0^1 S(\mu',\mu)\psi_2(a,\mu')\,d\mu' \quad , \quad \mu\epsilon(0,1) \quad , \tag{5}$$

where

$$S(\mu',\mu) = \frac{c_2\mu\mu'}{\mu+\mu'} H(\mu')H(\mu)[1 - \hat{c}(\mu+\mu') - l_2\mu\mu'] \quad , \tag{6}$$

with

$$l_2 = b_2(1 - c_2) \quad , \quad \hat{c} = \frac{c_2 l_2 H_1}{2 - c_2 H_0} \quad , \quad \text{and} \quad H_\alpha = \int_0^1 \mu^\alpha H(\mu)\,d\mu \quad , \tag{7}$$

to write the continuity condition, Eq. (2c), as

$$\psi_1(a,-\mu) = \frac{1}{2\mu} \int_0^1 S(\mu',\mu)\psi_1(a,\mu')\,d\mu' \quad , \quad \mu\epsilon(0,1) \quad . \tag{8}$$

We note that $H(\mu)$ required in the foregoing is the Chandrasekhar H function.[1] In addition to an analytical expression for $H(\mu)$, there exists the useful (for numerical calculations) nonlinear integral equation,

$$\frac{1}{H(\mu)} = 1 - \frac{c_2}{2}\, \mu \int_0^1 (1 + l_2 x^2) H(x) \frac{dx}{x+\mu} \quad , \quad \mu\epsilon[0,1] \quad . \tag{9}$$

If we now enter Eq. (3) into Eq. (8), we can use Eq. (9) and the recursive relation,

$$(2l+1)\mu P_l(\mu) = (l+1) P_{l+1}(\mu) + l P_{l-1}(\mu) \quad , \tag{10}$$

to evaluate the integration over μ' to obtain

$$H(\mu) \sum_{l=0}^{L} \sum_{j=1}^{[(L+1)/2]} \left(\frac{2l+1}{2}\right) \pi_l(-\mu) T_l(\nu_j) A_j \times [\exp(-a/\nu_j) + (-1)^l \exp(a/\nu_j)] = 0 \quad , \quad \mu\epsilon(0,1) \quad . \tag{11}$$

[1] S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, London and New York (1950).

[2] A. R. BURKART, Y. ISHIGURO, and C. E. SIEWERT, *Nucl. Sci. Eng.*, **61**, 72 (1976).

Here the π polynomials can readily be generated from

$$(2l + 1)\mu\pi_l(\mu) = K_l(-\mu) + (l + 1)\pi_{l+1}(\mu) + l\pi_{l-1}(\mu) \quad , \qquad (12)$$

with $\pi_0(\mu) = 1$ and

$$K_l(\mu) = \frac{c_2(2l + 1) \int_0^1 \mu' P_l(\mu') H(\mu') [1 - \hat{c}(\mu + \mu') - l_2\mu\mu'] d\mu'}{2 - c_2 H_0 + c_2\hat{c}H_1} \; . \qquad (13)$$

It is clear that the polynomials $K_l(\mu)$ can easily be expressed in terms of moments of $H(\mu)$. We note that for the case of isotropic scattering in the reflector, $b_2 = 0$, the polynomials $K_l(\mu)$ are simply constants. For the case $c_2 = 0$, the polynomials $K_l(\mu)$ are all zero; thus, $\pi_l(\mu) \to P_l(\mu)$, and Eq. (11) reduces to the statement that $\psi_1(a, -\mu) = 0$, $\mu \epsilon (0, 1)$, the obviously correct boundary condition for the bare slab.

For finite L, Eq. (11) cannot be satisfied pointwise, and so we choose to satisfy instead moments of Eq. (11):

$$\int_0^1 W(\mu) P_{2\alpha-1}(\mu) H(\mu) \sum_{l=0}^{L} \sum_{j=1}^{[(L+1)/2]} \left(\frac{2l + 1}{2}\right) \pi_l(-\mu) T_l(\nu_j) A_j \times [\exp(-a/\nu_j) + (-1)^l \exp(a/\nu_j)] d\mu = 0 \quad ,$$

$$\alpha = 1, 2, 3 \ldots \frac{L + 1}{2} \quad . \qquad (14)$$

In Eq. (14), the function $W(\mu)$ is arbitrary. For the vacuum case, $c_2 \to 0$, $H(\mu) \to 1$, $\pi_l(\mu) \to P_l(\mu)$, and thus if $W(\mu) = 1$, Eq. (14) will yield the traditional Marshak result.[3] In the present study, we have considered two choices for $W(\mu)$: $W(\mu) = H^{-1}(\mu)$ and $W(\mu) = 1$. In general, Eq. (14) is a system of algebraic equations, which we can write as

$$\mathbf{M}(a)\boldsymbol{A} = \mathbf{0} \quad , \qquad (15)$$

where $\boldsymbol{A}$ is a vector with elements A_α, $\alpha = 1, 2 \ldots [(L + 1)/2]$, and the $[(L + 1)/2]^2$ elements of the coefficient matrix $\mathbf{M}(a)$ are given by

$$M_{\alpha\beta}(a) = \sum_{l=0}^{L} \left(\frac{2l + 1}{2}\right) \int_0^1 W(\mu) P_{2\alpha-1}(\mu) H(\mu) \pi_l(-\mu) d\mu \, T_l(\nu_\beta) \times [\exp(-a/\nu_\beta) + (-1)^l \exp(a/\nu_\beta)] \quad . \qquad (16)$$

Clearly det $\mathbf{M}(a) = 0$ yields the desired half-thickness.

[3] K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts (1967).

NUMERICAL RESULTS

To tabulate concisely our numerical results, we let I $\Rightarrow$ traditional P-L solution, II $\Rightarrow$ Eq. (14) with $W(\mu) = H^{-1}(\mu)$,

TABLE I

Basic Data

Case	c_1	b_1	c_2	b_2
1	1.01	0.0	0.9	0.0
2	1.01	1.0	0.9	0.0
3	1.01	0.0	0.9	1.0
4	1.01	1.0	0.9	1.0
5	1.06	0.0	0.9	0.0
6	1.06	0.0	0.9	1.0
7	1.20	0.0	0.4	0.0
8	1.20	1.0	0.4	0.0
9	1.20	0.0	0.4	1.0
10	1.20	1.0	0.4	1.0
11	1.50	0.0	0.9	0.0
12	1.50	1.0	0.9	0.0
13	1.50	0.0	0.9	1.0
14	1.50	1.0	0.9	1.0
15	1.60	0.0	0.4	0.0
16	1.60	1.0	0.4	0.0
17	1.60	0.0	0.4	1.0
18	1.60	1.0	0.4	1.0

TABLE II

Critical Half-Thickness

	I			II			III				
Case	P-1	P-3	P-5	P-1	P-3	P-5	P-1	P-3	P-5	IV	Exact
1	7.301	7.2178	7.2157	7.287	7.2187	7.2160	7.266	7.2168	7.2154	7.214751	7.214751
2	8.510	8.3978	8.3949	8.490	8.3991	8.3953	8.459	8.3964	8.3945	8.393351	8.393532
3	7.576	7.4973	7.4957	7.563	7.4981	7.4960	7.545	7.4967	7.4956	7.499008	7.495017
4	8.909	8.8022	8.8000	8.891	8.8035	8.8004	8.863	8.8014	8.7998	8.804780	8.799010
5	2.149	2.0576	2.0538	2.139	2.0577	2.0539	2.125	2.0554	2.0532	2.052395	2.052360
6	2.347	2.2501	2.2470	2.338	2.2505	2.2471	2.324	2.2485	2.2465	2.249949	2.245784
7	1.352	1.2061	1.1894	1.301	1.1949	1.1855	1.296	1.1938	1.1852	1.182975	1.182419
8	1.550	1.3569	1.3371	1.478	1.3426	1.3321	1.472	1.3413	1.3317	1.328226	1.328237
9	1.391	1.2411	1.2251	1.342	1.2309	1.2216	1.337	1.2297	1.2212	1.221402	1.218626
10	1.607	1.4057	1.3867	1.536	1.3926	1.3822	1.529	1.3911	1.3818	1.382234	1.378425
11	0.3434	0.30030	0.29022	0.3408	0.29792	0.28854	0.3370	0.29545	0.28731	0.2910616	0.2825876
12	0.3537	0.30742	0.29684	0.3509	0.30492	0.29509	0.3468	0.30231	0.29381	0.3002338	0.2889746
13	0.4008	0.34412	0.33255	0.3979	0.34170	0.33088	0.3936	0.33881	0.32954	0.3345442	0.3248086
14	0.4173	0.35503	0.34273	0.4140	0.35245	0.34098	0.4091	0.34934	0.33956	0.3484569	0.3347018
15	0.5854	0.48585	0.46192	0.5520	0.47169	0.45519	0.5493	0.47048	0.45465	0.4509333	0.4468885
16	0.6539	0.52632	0.49844	0.6090	0.50942	0.49066	0.6056	0.50797	0.49003	0.4922023	0.4817374
17	0.6120	0.50455	0.48004	0.5787	0.49080	0.47362	0.5753	0.48929	0.47298	0.4706506	0.4653584
18	0.6907	0.55034	0.52155	0.6448	0.53375	0.51406	0.6403	0.53193	0.51331	0.5183184	0.5051470

TABLE III

Errors (in percent)

Case	I			II			III			IV
	P-1	P-3	P-5	P-1	P-3	P-5	P-1	P-3	P-5	
1	1.19	0.042	0.013	1.00	0.054	0.017	0.70	0.028	0.009	0.000
2	1.38	0.051	0.016	1.14	0.067	0.021	0.78	0.034	0.011	0.002
3	1.08	0.030	0.009	0.91	0.042	0.013	0.66	0.022	0.007	0.053
4	1.24	0.037	0.012	1.04	0.051	0.016	0.73	0.028	0.009	0.066
5	4.71	0.25	0.071	4.24	0.26	0.073	3.54	0.15	0.042	0.002
6	4.51	0.19	0.052	4.10	0.21	0.057	3.48	0.12	0.033	0.19
7	14.3	2.00	0.59	10.0	1.05	0.26	9.65	0.97	0.23	0.047
8	16.7	2.16	0.67	11.3	1.08	0.29	10.8	0.98	0.26	0.001
9	14.2	1.84	0.53	10.1	1.00	0.24	9.69	0.91	0.21	0.23
10	16.6	1.98	0.60	11.4	1.03	0.28	10.9	0.92	0.24	0.28
11	21.5	6.27	2.70	20.6	5.42	2.11	19.3	4.55	1.67	3.00
12	22.4	6.38	2.72	21.4	5.52	2.12	20.0	4.62	1.67	3.90
13	23.4	5.95	2.38	22.5	5.20	1.87	21.2	4.31	1.46	3.00
14	24.7	6.07	2.40	23.7	5.30	1.88	22.2	4.37	1.45	4.10
15	31.0	8.72	3.36	23.5	5.55	1.86	22.9	5.28	1.74	0.91
16	35.7	9.25	3.47	26.4	5.75	1.85	25.7	5.44	1.72	2.17
17	31.5	8.42	3.15	24.4	5.47	1.78	23.6	5.14	1.64	1.14
18	36.7	8.95	3.25	27.6	5.66	1.76	26.7	5.30	1.62	2.61

III ⇛ Eq. (14) with $W(\mu) = 1$, and IV ⇛ the "wide slab" approximation based on the extrapolated endpoint z_{0B}, as discussed in Ref. 2. Because the result is so simple in the P-1 approximation, we give

$$a^{\mathrm{I}} = \frac{1}{[(c_1 - 1)(3 - c_1 b_1)]^{1/2}} \tan^{-1}\left[\frac{(1 - c_2)^{1/2}(3 - c_1 b_1)^{1/2}}{(c_1 - 1)^{1/2}(3 - c_2 b_2)^{1/2}}\right] , \tag{17a}$$

$$a^{\mathrm{II}} = \frac{1}{[(c_1 - 1)(3 - c_1 b_1)]^{1/2}} \times \tan^{-1}\left\{\frac{(2 - c_2 H_0 + c_2 \hat{c} H_1)(3 - c_1 b_1)^{1/2}}{2(c_1 - 1)^{1/2}[H_1 \frac{3}{2} c_2 - H_2(c_2 l_2 + \frac{3}{2} c_2 \hat{c}) + 2 - c_2 H_0]}\right\} , \tag{17b}$$

and

$$a^{\mathrm{III}} = \frac{1}{[(c_1 - 1)(3 - c_1 b_1)]^{1/2}} \times \tan^{-1}\left\{\frac{(3 - c_1 b_1)^{1/2} H_1(2 - c_2 H_0 + c_2 \hat{c} H_1)}{3(c_1 - 1)^{1/2}[H_2(2 - c_2 H_0 - c_2 \hat{c} H_1) + c_2(H_1^2 - l_2 H_2^2)]}\right\} . \tag{17c}$$

Sets of selected data appear in Table I, and the results for approximations I, II, and III for $L = 1$, 3, and 5, along with approximation IV and the exact result,[2] are given in Table II. In Table III we give the errors associated with the various approximations for the cases considered.

Needless to say, the approximations given as I through IV are parameter dependent; however, we have observed generally that III is the best of the P - L-type approximations, as illustrated by Table III. Also, for hand calculations II or III is somewhat easier to handle than I, since the determinant of an $[(L + 1)/2] \times [(L + 1)/2]$ matrix is required, rather than the determinant of an $(L + 1) \times (L + 1)$ matrix. Of course, moments of the H function are required for II or III, but these moments are available.[4]

This method has been applied to the problem of a finite reflector, and the resulting expressions for the π polynomials (as well as explicit expressions for the case considered here) and some preliminary numerical results have been established.[4]

[4]K. NESHAT, MS Thesis, North Carolina State University, Raleigh, North Carolina (to be submitted).