Half-space analysis basic to the time-dependent BGK model in the kinetic theory of gases

C. E. Siewert and E. E. Burniston

Departments of Nuclear Engineering and Mathematics, North Carolina State University, Raleigh, North Carolina 27607
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The elementary solutions of the linearized time-dependent BGK equation are shown to have, for the case of no discrete eigenvalues, the half-range expansion property necessary for half-space analysis. Also the partial indices corresponding to the basic matrix Riemann problem encountered are shown, for the general case, to be nonnegative, as required for the half-space analysis.

I. INTRODUCTION

The time-dependent BGK model in the kinetic theory of gases can be linearized and expressed in the manner

$$\left(\frac{\partial}{\partial t}+c_x\frac{\partial}{\partial x}+1\right)h(x,\mathbf{c},t)=(\pi)^{-3/2}\int_{E_3} h(x,\mathbf{c}',t)[1+2\mathbf{c}\cdot\mathbf{c}'+\tfrac{2}{3}(c'^2-\tfrac{3}{2})(c^2-\tfrac{3}{2})]e^{-c'^2}d^3c', \quad (1)$$

where $h(x,\mathbf{c},t)$ represents the perturbation of the distribution from the Maxwellian distribution, $\mathbf{c}$, with components c_x, c_y, and c_z and magnitude c, is the velocity, t is the time, and x is the space variable. In the manner of Cercignani,[1] we find that Eq. (1) can be decomposed, by taking moments, into a set of two coupled equations plus three uncoupled equations. Since the uncoupled equations have been discussed in considerable detail,[1] we consider here only the coupled equations,

$$\left(\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}+1\right)\Psi(x,\mu,t)=(\pi)^{-1/2}\int_{-\infty}^{\infty}[\mathbf{Q}(\mu)\tilde{\mathbf{Q}}(\mu')+\mathbf{P}(\mu)\tilde{\mathbf{P}}(\mu')]\Psi(x,\mu',t)\,e^{-\mu'^2}d\mu'. \quad (2)$$

Here, the elements of the two-vector $\Psi(x,\mu,t)$ are related[1] to the density and temperature of the gas, and x, μ, and t represent, respectively, the position, velocity component, and time, in dimensionless units. In addition,

$$\mathbf{Q}(\mu)=\begin{bmatrix}(\frac{2}{3})^{1/2}(\mu^2-\frac{1}{2}) & 1\\ (\frac{2}{3})^{1/2} & 0\end{bmatrix} \quad (3a)$$

and

$$\mathbf{P}(\mu)=(2)^{1/2}\mu\begin{matrix}1 & 0\\ 0 & 0\end{matrix} \quad (3b)$$

We note that the time-independent version of Eq. (2) has been studied extensively by Kriese, Chang, and Siewert.[2]

Since we wish ultimately to solve initial and boundary-value problems relevant to Eq. (2), we first will establish the required elementary solutions.

II. ELEMENTARY SOLUTIONS

We seek solutions of Eq. (2) of the form

$$\Psi(x,\mu,t)=\exp(st)\Phi(\nu,\mu;s)\exp[-(s+1)x/\nu], \quad (4)$$

where, in general, s is complex, but $s\neq-1$, and ν is to be determined. Equation (4) can be entered into Eq. (2) to yield, after some elementary analysis has been carried out,

$$(\nu-\mu)\Phi(\nu,\mu;s)=\omega\nu\mathbf{Q}(\mu)[\mathbf{I}+\gamma\nu\mu\mathbf{D}]\mathbf{M}(\nu;s), \quad (5)$$

where

$$\gamma=2s/(s+1), \quad (6a)$$

$$(\pi)^{1/2}\omega=1/(s+1), \quad (6b)$$

$$\mathbf{D}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \quad (7)$$

and the normalization vector is given by

$$\mathbf{M}(\nu;s)=\int_{-\infty}^{\infty}\tilde{\mathbf{Q}}(\mu)\Phi(\nu,\mu;s)\,e^{-\mu^2}d\mu. \quad (8)$$

Since the velocity component $\mu\in(-\infty,\infty)$, we can solve Eq. (5) for $\nu\in(-\infty,\infty)$ by writing

$$\Phi(\nu,\mu;s)=\omega\left[\nu Pv\left(\frac{1}{\nu-\mu}\right)+\hat{\lambda}(\nu)\,\delta(\nu-\mu)\right]\mathbf{Q}(\mu)\times(\mathbf{I}+\gamma\nu\mu\,\mathbf{D})\mathbf{M}(\nu;s). \quad (9)$$

Here $Pv(1/x)$ denotes the Cauchy principal-value distribution, and $\delta(x)$ represents the Dirac delta distribution. We note that Eq. (9) is a generalization of the "singular eigenfunction" introduced in 1960 by Case[3] and discussed extensively in the text by Case and Zweifel.[4] In Eq. (9) the function $\hat{\lambda}(\nu)$ is considered, at this point, "arbitrary"; however, if we multiply Eq. (9) by $\tilde{\mathbf{Q}}(\mu)\exp(-\mu^2)$ and integrate over μ, we find

$$[\boldsymbol{\lambda}(\nu;s)-\hat{\lambda}(\nu)\boldsymbol{\Psi}(\nu;s)]\mathbf{M}(\nu;s)=\mathbf{0}, \quad (10)$$

where

$$\boldsymbol{\Psi}(\mu;s)=\omega e^{-\mu^2}\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu)(\mathbf{I}+\gamma\mu^2\mathbf{D}), \quad (11)$$

and

$$\boldsymbol{\lambda}(\mu;s)=\mathbf{I}+\mu P\int_{-\infty}^{\infty}\boldsymbol{\Psi}(\nu;s)\frac{d\nu}{\nu-\mu}. \quad (12)$$

From Eq. (10) we deduce that $\det[\boldsymbol{\lambda}(\nu;s)-\hat{\lambda}(\nu)\boldsymbol{\Psi}(\nu;s)]=0$ and hence that there exist two $\hat{\lambda}$'s, i.e., $\hat{\lambda}_1(\nu)$ and $\hat{\lambda}_2(\nu)$. We thus write our so-called continuum solutions as

$$\Phi_\alpha(\nu,\mu;s)=\omega\left[\nu Pv\left(\frac{1}{\nu-\mu}\right)+\hat{\lambda}_\alpha(\nu)\,\delta(\nu-\mu)\right]\times\mathbf{Q}(\mu)(\mathbf{I}+\gamma\nu\mu\,\mathbf{D})\mathbf{M}_\alpha(\nu;s), \quad \alpha=1 \text{ and } 2. \quad (13)$$

In regard to the discrete spectrum, we consider now $\nu \notin (-\infty, \infty)$ and write

$$\mathbf{\Phi}(\pm \nu_\alpha, \mu; s) = \omega\left(\frac{\nu_\alpha}{\nu_\alpha \mp \mu}\right)\mathbf{Q}(\mu)(\mathbf{I} \pm \gamma\nu_\alpha\mu\mathbf{D})\mathbf{M}(\nu_\alpha; s), \tag{14}$$

where

$$\Lambda(\pm \nu_\alpha; s)\mathbf{M}(\nu_\alpha; s) = \mathbf{0}, \tag{15}$$

$$\Lambda(z; s) = \mathbf{I} + z\int_{-\infty}^{\infty}\Psi(\mu; s)\frac{d\mu}{\mu - z}, \tag{16}$$

and ν_α is used to denote each of the "positive" zeros of $\Lambda(z; s) = \det\mathbf{\Lambda}(z; s)$.

As we have discussed in a previous paper,[5] the dispersion function can be written as

$$\begin{aligned}\Lambda(z; s) = \frac{1}{(s+1)^3}\{&\tfrac{1}{3}s^2z^2 + (s+1)(s-\tfrac{1}{3})(s+\tfrac{1}{2}) \\ &+ [\tfrac{2}{3}s^2z^4 + \tfrac{1}{3}z^2(4s^2-1) + \tfrac{1}{2}(s+1)(\tfrac{11}{3}s+1)]\Lambda(z) \\ &+ \tfrac{2}{3}(s+1+2s^2)\Lambda^2(z)\},\end{aligned} \tag{17}$$

where

$$\Lambda(z) = 1 + \frac{1}{\sqrt{\pi}}z\int_{-\infty}^{\infty}e^{-\mu^2}\frac{d\mu}{\mu - z}. \tag{18}$$

We have shown[5] that $\Lambda(z; s)$ has $\kappa(s)$ pairs of zeros, where $\kappa(s)$ can be either 0, 1, 2, or 3 when s is contained respectively in S_0, S_1, S_2, or S_3, as previously defined.[5]

Having established the required elementary solutions of Eq. (2), we now formally write our general solution (with s as a parameter) as

$$\begin{aligned}\Psi(x, \mu, t; s) = e^{st}\Big\{&\sum_{\alpha=1}^{\kappa(s)}[A(\nu_\alpha)\mathbf{\Phi}(\nu_\alpha, \mu; s)\exp[-(s+1)x/\nu_\alpha] \\ &+ A(-\nu_\alpha)\mathbf{\Phi}(-\nu_\alpha, \mu; s)\exp[(s+1)x/\nu_\alpha]] \\ &+ \int_{-\infty}^{\infty}\sum_{\alpha=1}^{2}A_\alpha(\nu)\mathbf{\Phi}_\alpha(\nu, \mu; s) \\ &\times \exp[-(s+1)x/\nu]\,d\nu\Big\}.\end{aligned} \tag{19}$$

Here $A(\pm\nu_\alpha)$, $A_1(\nu)$, and $A_2(\nu)$ are the expansion coefficients to be determined from the boundary and initial conditions. If we let $\mathbf{A}(\nu)$ denote the expansion vector

$$\mathbf{A}(\nu) = A_1(\nu)\mathbf{M}_1(\nu; s) + A_2(\nu)\mathbf{M}_2(\nu; s), \tag{20}$$

then Eq. (19) can be written as

$$\begin{aligned}\Psi(x, \mu, t; s) = e^{st}\Big\{&\sum_{\alpha=1}^{\kappa(s)}[A(\nu_\alpha)\mathbf{\Phi}(\nu_\alpha, \mu; s)\exp[-(s+1)x/\nu_\alpha] \\ &+ A(-\nu_\alpha)\mathbf{\Phi}(-\nu_\alpha, \mu; s)\exp[(s+1)x/\nu_\alpha]] \\ &+ \int_{-\infty}^{\infty}\mathbf{\Phi}(\nu, \mu; s)\mathbf{A}(\nu)\exp[-(s+1)x/\nu]\,d\nu\Big\},\end{aligned} \tag{21}$$

where the continuum *matrix* is

$$\begin{aligned}\mathbf{\Phi}(\nu, \mu; s) = &\ \omega\nu P\nu\left(\frac{1}{\nu - \mu}\right)\mathbf{Q}(\mu)(\mathbf{I} + \gamma\nu\mu\mathbf{D}) \\ &+ \delta(\nu - \mu)e^{\nu^2}\tilde{\mathbf{Q}}^{-1}(\nu)\boldsymbol{\lambda}(\nu; s).\end{aligned} \tag{22}$$

III. HALF-RANGE ANALYSIS

We wish now to show that the "eigenvectors" established in the previous section have an important property that allows us to write

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{\kappa(s)}A(\nu_\alpha)\mathbf{\Phi}(\nu_\alpha, \mu; s) + \int_0^\infty\mathbf{\Phi}(\nu, \mu; s)\mathbf{A}(\nu)\,d\nu,$$
$$\mu \in (0, \infty), \tag{23}$$

where $\mathbf{I}(\mu)$ is an arbitrary two-vector which is Hölder continuous on any bounded interval of the positive real axis and further satisfies

$$|I_\alpha(\mu)| < Ce^\mu, \quad \alpha = 1, 2, \quad \mu \in (0, \infty), \tag{24}$$

where C is a positive constant.

Equation (23) is the statement equivalent to Case's[3] half-range completeness theorem for the one-speed neutron problem and clearly will be required when we wish to solve explicitly a typical half-space problem.

In order to illustrate explicitly the analysis required to prove Eq. (23), we consider currently only those values of $s \in S_0$, so we can allow $\kappa(s)$ to be zero. By introducing the sectionally analytic vector function

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_0^\infty\nu(\mathbf{I} + \gamma\nu z\mathbf{D})\mathbf{A}(\nu)\frac{d\nu}{\nu - z}, \tag{25}$$

with limiting values

$$\begin{aligned}\mathbf{N}^\pm(t) = &\ \frac{1}{2\pi i}P\int_0^\infty\nu(\mathbf{I} + \gamma\nu t\mathbf{D})\mathbf{A}(\nu)\frac{d\nu}{\nu - t} \\ &\pm \tfrac{1}{2}t(\mathbf{I} + \gamma t^2\mathbf{D})\mathbf{A}(t),\end{aligned} \tag{26}$$

we can express the equation

$$\mathbf{I}(\mu) = \int_0^\infty\mathbf{\Phi}(\nu, \mu; s)\mathbf{A}(\nu)\,d\nu, \quad \mu \in (0, \infty) \text{ and } s \in S_0, \tag{27}$$

in the form

$$\begin{aligned}\mu\mathbf{\Pi}(\mu)\tilde{\mathbf{Q}}(\mu)e^{-\mu^2}\mathbf{I}(\mu) = &\ \mathbf{\Omega}^+(\mu; s)\Pi^{-1}(-\mu)\mathbf{N}^+(\mu) \\ &- \mathbf{\Omega}^-(\mu; s)\Pi^{-1}(-\mu)\mathbf{N}^-(\mu).\end{aligned} \tag{28}$$

Here we have introduced

$$\Pi(z) = \mathbf{I} - \left(\frac{z}{z_1}\right)\mathbf{D}, \quad (\gamma)^{1/2}z_1 = i, \tag{29}$$

and

$$\mathbf{\Omega}^\pm(\mu; s) = \Pi(\mu)[\boldsymbol{\lambda}(\mu; s) \pm \pi i\mu\Psi(\mu; s)]\Pi^{-1}(\mu). \tag{30}$$

It can easily be shown that the matrix

$$\mathbf{\Omega}(z; s) = \mathbf{\Pi}(z)\Lambda(z; s)\mathbf{\Pi}^{-1}(z) \tag{31}$$

has the limiting values given by Eq. (30) and can be written as

$$\mathbf{\Omega}(z; s) = \mathbf{I} + z\int_{-\infty}^{\infty}\hat{\Psi}(\mu; s)\frac{d\mu}{\mu - z}, \tag{32}$$

where

$$\hat{\Psi}(\mu; s) = \Pi(\mu)\Psi(\mu; s)\Pi^{-1}(\mu). \tag{33}$$

The $\mathbf{\Omega}(z; s)$ matrix has other important properties that we will soon require,

$$\overline{\mathbf{\Omega}(\bar{z}; \bar{s})} \equiv \bar{\bar{\mathbf{\Omega}}}(z; s) = \mathbf{\Omega}(-z; s) = \tilde{\mathbf{\Omega}}(z; s). \tag{34}$$

If now we let $\mathbf{X}(z; s)$ denote a canonical solution, of ordered normal form at infinity,[5,6] to the matrix Riemann problem defined by

$$\mathbf{X}^{+}(\mu;s)=\mathbf{G}(\mu;s)\,\mathbf{X}^{-}(\mu;s),\quad \mu\in[0,\infty), \tag{35}$$

where

$$\mathbf{G}(\mu;s)=\tilde{\mathbf{\Omega}}^{+}(\mu;s)[\tilde{\mathbf{\Omega}}^{-}(\mu;s)]^{-1}, \tag{36}$$

then we can write the solution to Eq. (28) as

$$\tilde{\mathbf{X}}(z;s)\,\mathbf{\Pi}^{-1}(-z)\,\mathbf{N}(z)=\frac{1}{2\pi i}\left(\int_0^\infty \mathbf{\Gamma}(\mu)\frac{d\mu}{\mu-z}+R(z)\right), \tag{37}$$

where

$$\mathbf{\Gamma}(\mu)=\mu\tilde{\mathbf{X}}^{+}(\mu;s)\,[\mathbf{\Omega}^{+}(\mu;s)]^{-1}\,\mathbf{\Pi}(\mu)\tilde{\mathbf{Q}}(\mu)\,e^{-\mu^2}\mathbf{I}(\mu). \tag{38}$$

In Eq. (37) we use $\mathbf{R}(z)$ to denote a vector of rational functions. At this point we wish to make use of the fact (proved in the next section) that the partial indices, κ_1 and κ_2, basic to the Riemann problem defined by Eqs. (35) and (36), are nonnegative. Since we are considering here the case $\kappa=0$, then clearly $\kappa_1=\kappa_2=\kappa=0$, and thus we can normalize our canonical solution by taking

$$\mathbf{X}(\infty;s)=\mathbf{I},\quad \kappa=0. \tag{39}$$

On investigating Eq. (37) for large z and noting that $\mathbf{R}(z)$ can be singular only at $z=-z_1$, we conclude, after examining the form of Eq. (25) for large $|z|$, that

$$\mathbf{R}(z)=(z_1+z)^{-1}\,\mathbf{R}, \tag{40}$$

where the constant vector $\mathbf{R}$ can be expressed as

$$\mathbf{R}=\int_0^\infty \mathbf{\Gamma}(\mu)\,d\mu-(\mathbf{I}-\mathbf{D})\int_0^\infty \nu\mathbf{A}(\nu)\,d\nu-\gamma z_1\mathbf{D}\int_0^\infty \mathbf{A}(\nu)\nu^2\,d\nu. \tag{41}$$

Thus we can now write Eq. (37) as

$$\mathbf{\Pi}^{-1}(-z)\,\mathbf{N}(z)=\tilde{\mathbf{X}}^{-1}(z;s)\frac{1}{2\pi i}\left[\int_0^\infty \mathbf{\Gamma}(\mu)\frac{d\mu}{\mu-z}+\frac{1}{z+z_1}\,\mathbf{R}\right]. \tag{42}$$

If now we notice that Eq. (26) yields

$$\mathbf{N}^{+}(t)-\mathbf{N}^{-}(t)=t\mathbf{\Pi}(t)\,\mathbf{\Pi}(-t)\,\mathbf{A}(t), \tag{43}$$

we can obtain from Eq. (42) the expression

$$t\mathbf{\Pi}(t)\,\mathbf{A}(t)=\int_0^\infty\left(\mathbf{U}(t)\frac{P}{\mu-t}+\mathbf{V}(t)\,\delta(\mu-t)\right)\mathbf{X}(-t;s)\,\mathbf{\Omega}(\infty;s)\times\mathbf{\Gamma}(\mu)\,d\mu+\mathbf{U}(t)\,\mathbf{X}(-t;s)\,\mathbf{\Omega}(\infty;s)\frac{\mathbf{R}}{z_1+t}. \tag{44}$$

In developing Eq. (44) we have used the fact that the $\mathbf{X}$ matrix factors $\mathbf{\Omega}(z;s)$ in the following manner:

$$\tilde{\mathbf{\Omega}}(z;s)=\mathbf{X}(z;s)\,\mathbf{\Omega}(\infty;s)\,\tilde{\mathbf{X}}(-z;s),\quad \kappa=0. \tag{45}$$

Here we have defined

$$2\pi i\mathbf{U}(t)=[\mathbf{\Omega}^{+}(t;s)]^{-1}-[\mathbf{\Omega}^{-}(t;s)]^{-1} \tag{46a}$$

and

$$2\mathbf{V}(t)=[\mathbf{\Omega}^{+}(t;s)]^{-1}+[\mathbf{\Omega}^{-}(t;s)]^{-1}. \tag{46b}$$

We can find from Eq. (44) the moments of $\mathbf{A}(\nu)$ required in Eq. (41) to establish $\mathbf{R}$. After using the integral representation

$$\mathbf{\Omega}^{-1}(z;s)\,\mathbf{X}(-z;s)=\mathbf{\Omega}^{-1}(\infty;s)+\int_0^\infty \mathbf{U}(t)\,\mathbf{X}(-t;s)\frac{dt}{t-z},\quad \kappa=0, \tag{47}$$

to help simplify our result, we find

$$\mathbf{R}=-2z_1[\mathbf{I}+\tilde{\mathbf{X}}(-z_1;s)\mathbf{D}\tilde{\mathbf{X}}^{-1}(z_1;s)]^{-1}\times\tilde{\mathbf{X}}(-z_1;s)\mathbf{D}\tilde{\mathbf{X}}^{-1}(z_1;s)\int_0^\infty \mathbf{\Gamma}(\mu)\frac{d\mu}{\mu-z_1}. \tag{48}$$

Since $\mathbf{R}$ is now established explicitly, we consider the proof that $\mathbf{I}(\mu)$ can be represented as given by Eq. (23) completed for the case $\kappa=0$.

IV. PARTIAL INDICES

As mentioned in the previous section, we require in our proof of "half-range completeness" the knowledge that the partial indices basic to the matrix Riemann problem defined by Eqs. (35) and (36) are nonnegative. The proof that we will develop here is similar to the one given previously[7] for a problem relating to the scattering of polarized light; however, because $\mathbf{\Omega}(z;s)$ is not symmetric and because the problem contains a complex parameter s, some additional work is required. We consider in this section the total index κ to be 0, 1, 2, or 3.

First of all, we note that

$$\mathbf{\Phi}(z;s)\equiv\tilde{\mathbf{\Omega}}(z;s)\,\tilde{\mathbf{X}}^{-1}(-z;s) \tag{49}$$

is a solution of the Riemann problem defined by

$$\mathbf{\Phi}^{+}(\mu;s)=\mathbf{G}(\mu;s)\,\mathbf{\Phi}^{-}(\mu;s),\quad \mu\in[0,\infty), \tag{50}$$

where $\mathbf{G}(\mu;s)$ is given by Eq. (36), and thus[6] $\mathbf{\Phi}(z;s)$ can be expressed as

$$\mathbf{\Phi}(z;s)=\mathbf{X}(z;s)\,\mathbf{P}(z), \tag{51}$$

where $\mathbf{P}(z)$ is a matrix of polynomials. It is clear that Eqs. (49) and (51) yield the factorization

$$\tilde{\mathbf{\Omega}}(z;s)=\mathbf{X}(z;s)\,\mathbf{P}(z)\,\tilde{\mathbf{X}}(-z;s). \tag{52}$$

We note that by definition[6] a canonical solution of ordered normal form at infinity is such that

$$\lim_{|z|\to\infty}\mathbf{X}(z;s)\begin{bmatrix} z^{\kappa_1} & 0\\ 0 & z^{\kappa_2}\end{bmatrix}=\mathbf{K},\ \det\mathbf{K}\neq 0, \tag{53}$$

where $\kappa_1\leqslant\kappa_2$ and κ_2 are the partial indices and $\kappa_1+\kappa_2=\kappa$. If we use Eq. (53) in Eq. (52), as $|z|\to\infty$, we can readily deduce that $\kappa_1\geqslant 0$ unless $P_{11}(z)\equiv 0$. Thus to show that $\kappa_2\geqslant\kappa_1\geqslant 0$, we need to prove that $P_{11}(z)\neq 0$.

If we now change s to $\bar{s}$ in Eqs. (35) and (36) and take the complex conjugate of the resulting equations, we can use Eq. (34) to deduce that

$$\bar{\bar{\mathbf{X}}}^{+}(\mu;s)=\mathbf{G}_*(\mu;s)\bar{\bar{\mathbf{X}}}^{-}(\mu;s),\quad \mu\in[0,\infty), \tag{54}$$

where

$$\mathbf{G}_*(\mu;s)=[\overline{\mathbf{G}(\mu;\bar{s})}]^{-1}=\mathbf{\Omega}^{+}(\mu;s)[\mathbf{\Omega}^{-}(\mu;s)]^{-1} \tag{55}$$

and we have defined

$$\bar{\bar{\mathbf{X}}}(z;s)=\overline{\mathbf{X}(\bar{z};\bar{s})}. \tag{56}$$

Using the fact that $\mathbf{W}(z;s)=\mathbf{\Pi}(z)\,\mathbf{\Pi}(-z)\,\mathbf{\Lambda}(z;s)$ is symmetric, we can deduce a convenient relationship,

$$\mathbf{B}(\mu)\,\mathbf{G}_*(\mu;s)=\mathbf{G}(\mu;s)\,\mathbf{B}(\mu), \tag{57}$$

where

$$\mathbf{B}(\mu)=\mathbf{\Pi}^{-1}(\mu)\,\mathbf{\Pi}(-\mu). \tag{58}$$

It is not difficult now to deduce that

$$\bar{\bar{\mathbf{X}}}(z;s)=\mathbf{B}^{-1}(z)\,\mathbf{X}(z;s)\,\mathcal{R}(z), \tag{59}$$

where $\mathcal{R}(z)$ is a matrix of rational functions. Further, we observe from Eq. (59) that $\mathcal{R}(z)$ must be of the form

$$\mathcal{R}(z)=\left(\frac{1}{z_1-z}\right)\hat{\mathcal{P}}(z), \tag{60}$$

where $\hat{\mathcal{P}}(z)$ has polynomial elements. If we let

$$\mathbf{X}(z;s)\to\mathbf{K}(s)\begin{bmatrix} z^{-\kappa_1} & 0\\ 0 & z^{-\kappa_2}\end{bmatrix},\quad |z|\to\infty, \tag{61}$$

then we can find the general form of $\hat{\mathcal{P}}(z)$ by investigating Eqs. (59) and (60) as $|z|\to\infty$:

$$\hat{\mathcal{P}}(z)\to z\begin{bmatrix} T_{11} & T_{12}\;z^{\kappa_1-\kappa_2}\\ T_{21}\;z^{\kappa_2-\kappa_1} & T_{22}\end{bmatrix},\quad |z|\to\infty, \tag{62}$$

where

$$\mathbf{T}=\mathbf{K}^{-1}(s)\begin{bmatrix} -1 & 0\\ 0 & 1\end{bmatrix}\overline{\mathbf{K}(\bar{s})}. \tag{63}$$

We assume here that $\kappa_1\neq\kappa_2$, for otherwise no proof that κ_1 and κ_2 are nonnegative is required, and thus we can consider

$$\mathbf{K}(s)=\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix} \tag{64a}$$

or

$$\mathbf{K}(s)=\begin{bmatrix} 1 & b(s)\\ 0 & 1\end{bmatrix} \tag{64b}$$

and deduce that the most general form of $\hat{\mathcal{P}}(z)$ is

$$\hat{\mathcal{P}}(z)=\pm\begin{bmatrix} \hat{P}_{11}+z & 0\\ \hat{P}_{21}(z) & \hat{P}_{22}-z\end{bmatrix}, \tag{65}$$

where $\hat{P}_{11}$ and $\hat{P}_{22}$ are constants. If now we use Eqs. (65) and (60) and evaluate Eq. (59) at $z=0$, we obtain

$$\overline{X_{12}(0;\bar{s})}=\pm\left(\frac{1}{z_1}\right)\hat{P}_{22}\,X_{12}(0;s) \tag{66a}$$

and

$$\overline{X_{22}(0;\bar{s})}=\pm\left(\frac{1}{z_1}\right)\hat{P}_{22}\,X_{22}(0;s). \tag{66b}$$

Equations (66) allow us to prove the required statement that the polynomial $P_{11}(z)$ appearing in Eq. (52) is not identically zero. Since $\mathbf{\Omega}(0;s)=\mathbf{I}$, we can solve Eq. (52) to obtain

$$P_{11}(0)=X^{-2}(0;s)\,[X_{12}^2(0;s)+X_{22}^2(0;s)], \tag{67}$$

where $X(0;s)=\det\mathbf{X}(0;s)$. To allow $P_{11}(0)=0$ yields

$$X_{22}(0;s)=\pm iX_{12}(0;s) \tag{68}$$

which contradicts Eqs. (66). Since $P_{11}(0)\neq 0$ it follows that the partial indices basic to the Riemann problem defined by Eqs. (35) and (36) are nonnegative.

V. THE H MATRIX

If we go back to Eq. (52) and use the normalization $\mathbf{X}(\infty;s)=\mathbf{I}$, $\kappa=0$, we can write

$$\tilde{\mathbf{\Omega}}(z;s)=\mathbf{X}(z;s)\,\mathbf{X}^{-1}(0;s)\,\tilde{\mathbf{X}}^{-1}(0;s)\,\tilde{\mathbf{X}}(-z;s). \tag{69}$$

Therefore, if we define the $\mathbf{H}$ matrix by

$$\mathbf{H}(z;s)=\tilde{\mathbf{X}}^{1}(-z;s)\,\tilde{\mathbf{X}}(0;s), \tag{70}$$

then a factorization of $\mathbf{\Omega}(z;s)$ becomes

$$\tilde{\mathbf{\Omega}}(z;s)=\tilde{\mathbf{H}}^{-1}(-z;s)\,\mathbf{H}^{-1}(z;s). \tag{71}$$

Since ultimately we wish to express all of our results in terms of the convenient $\mathbf{H}$ matrix, we can use

$$\mathbf{H}^{-1}(z;s)=\mathbf{H}^{-1}(\infty;s)+\frac{1}{2\pi i}\times\int_{-\infty}^{0}\tilde{\mathbf{H}}(-t;s)\,[\tilde{\mathbf{\Omega}}^{+}(t;s)-\tilde{\mathbf{\Omega}}^{-}(t;s)]\,\frac{dt}{t-z} \tag{72}$$

or

$$\mathbf{H}^{-1}(z;s)=\mathbf{I}-z\int_0^\infty\tilde{\mathbf{H}}(t;s)\,\hat{\mathbf{\Psi}}(t;s)\frac{dt}{t+z} \tag{73}$$

to compute $\mathbf{H}(z;s)$ for $z\notin(0,\infty)$ after we have solved

$$\mathbf{H}^{-1}(\mu;s)=\mathbf{I}-\mu\int_0^\infty\tilde{\mathbf{H}}(t;s)\,\hat{\mathbf{\Psi}}(t;s)\,\frac{dt}{t+\mu},\quad \mu\in[0,\infty), \tag{74}$$

iteratively. It is clear that Eq. (74) has a solution since we know that $\mathbf{X}(z;s)$ exists and the subsequent definition of $\mathbf{H}(z;s)$ in terms of $\mathbf{X}(z;s)$; however, the recent work of Zweifel and co-workers[8,9] could prove very useful for showing that an iterative solution of Eq. (74) converges to the desired result.

VI. SOUND-WAVE PROPAGATION

It is evident that we can readily solve half-space problems based on Eq. (2) subject to a free-surface boundary condition of the form

$$\mathbf{\Psi}(0,\mu,t)=\exp(st)\,\mathbf{F}(\mu),\quad \mu\geq 0, \tag{75}$$

and a specified condition as $x\to\infty$. Here, we consider $\mathbf{F}(\mu)$ to be, in general, an arbitrary Hölder function. For example, for sound-wave propagation in a half-space defined by $\Psi(x,\mu,t)\to 0$ as $x\to\infty$, and

$$\mathbf{\Psi}(0,\mu,t)=\exp(i\omega t)\,\mathbf{F}(\mu),\quad \mu>0, \tag{76}$$

we simply let $s=i\omega$ and write the desired solution as

$$\mathbf{\Psi}(x,\mu,t)=\exp(i\omega t)\left[\sum_{\alpha=1}^{\kappa}A(\nu_\alpha)\,\Phi(\nu_\alpha,\mu;i\omega)\exp[-(i\omega+1)x/\nu_\alpha]+\int_0^\infty\mathbf{\Phi}(\nu,\mu;i\omega)A(\nu)\exp[-(i\omega+1)x/\nu]\,d\nu\right]. \tag{77}$$

If we constrain Eq. (77) to meet Eq. (76), we get

$$\mathbf{F}(\mu)=\sum_{\alpha=1}^{\kappa}A(\nu_\alpha)\,\Phi(\nu_\alpha,\mu;i\omega)+\int_0^\infty\Phi(\nu,\mu;i\omega)\,\mathbf{A}(\nu)\,d\nu,\quad \mu\geq 0. \tag{78}$$

The solution of Eq. (78) is given in Sec. III for the case $\kappa=0$; we note from our previous work[5] that $\omega>2.14517\cdots\Longrightarrow\kappa=0$. We are confident that explicit solutions of Eq. (78) for a general index will soon be forthcoming.

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