Half-space Analysis Basic to the Linearized Boltzmann Equation

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1. Introduction

We wish to consider here the linearized Boltzmann equation written, for steady state conditions, as [1]

$$\left(c_1 \frac{\partial}{\partial x_1} + 1\right) h(x_1, c_1, c_2, c_3) = \int h(x_1, c_1', c_2', c_3') K(\mathbf{c}':\mathbf{c})\, e^{-c'^2}\, d^3c'. \tag{1}$$

Here x_1, c_1, c_2 and c_3 are, respectively, the non-dimensional space variable and velocity components, $\mathbf{c}$ is the velocity of the particles and $c = |\mathbf{c}|$. The dependent variable h represents the perturbation of the particle distribution function from the Maxwellian [1]. In addition, the scattering kernel is taken here to be one corresponding to the linearized BGK model,

$$K(\mathbf{c}':\mathbf{c}) = \frac{1}{\pi^{3/2}} [1 + 2\mathbf{c}'\cdot\mathbf{c} + \tfrac{2}{3}(c'^2 - \tfrac{3}{2})(c^2 - \tfrac{3}{2})]. \tag{2}$$

Since we are interested here in temperature-density effects, we can take 'moments' of Eqn. (1) to obtain equations dependent only on x_1 and c_1. Thus we let

$$\Psi_1(x_1, c_1) = \pi^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(c_2^2+c_3^2)}\, h(x_1, c_1, c_2, c_3)\, dc_2\, dc_3 \tag{3a}$$

and

$$\Psi_2(x_1, c_1) = \pi^{-1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(c_2^2+c_3^2)}\, h(x_1, c_1, c_2, c_3)(c_2^2 + c_3^2 - 1)\, dc_2\, dc_3 \tag{3b}$$

so that the density perturbation

$$\Delta N(x_1) = \pi^{-3/2} \int h(x_1, c_1, c_2, c_3)\, e^{-c^2}\, d^3c \tag{4}$$

and the temperature perturbation

$$\Delta T(x_1) = \tfrac{2}{3}\pi^{-3/2} \int h(x_1, c_1, c_2, c_3)(c^2 - \tfrac{3}{2})\, e^{-c^2}\, d^3c \tag{5}$$

can be expressed as

$$\Delta N(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Psi_1(x, \mu)\, e^{-\mu^2}\, d\mu \tag{6}$$

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and

$$\Delta T(x) = \frac{2}{3\pi} \int_{-\infty}^{\infty} [(\mu^2 - \tfrac{1}{2})\Psi_1(x, \mu) + \Psi_2(x, \mu)]\, e^{-\mu^2}\, d\mu, \tag{7}$$

where we have used x for x_1 and μ for c_1. If now we integrate Eqn. (1) from $-\infty$ to ∞ over both c_2 and c_3 and then multiply Eqn. (1) by $(c_2^2 + c_3^2 - 1)$ and integrate similarly, we find that the resulting two coupled equations can be written as

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Psi}(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [\mathbf{Q}(\mu)\tilde{\mathbf{Q}}(\mu') + 2\mu\mu'\mathbf{P}]\mathbf{\Psi}(x, \mu')\, e^{-\mu'^2}\, d\mu', \tag{8}$$

where $\mathbf{\Psi}(x, \mu)$ is a two-vector with elements $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$,

$$\mathbf{Q}(\mu) = \begin{vmatrix} (\tfrac{2}{3})^{1/2}(\mu^2 - \tfrac{1}{2}) & 1 \\ (\tfrac{2}{3})^{1/2} & 0 \end{vmatrix} \quad \text{and} \quad \mathbf{P} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}. \tag{9}$$

We note that we can deduce from Eqn. (8) that $\mathbf{P}\mathbf{J}_1(x)$, where

$$\mathbf{J}_1(x) = \int_{-\infty}^{\infty} \mu\, e^{-\mu^2}\, \mathbf{\Psi}(x, \mu)\, d\mu, \tag{10}$$

is a constant and thus this term can effectively be removed [2] from the equation to give the equation studied by Kriese, Chang and Siewert [3] in a paper hereafter referred to as KCS.

2. Half-Space Problems

We note that the elementary solutions of Eqn. (8) and the required half-range completeness and orthogonality theorems concerning the elementary solutions have been reported in KCS. Here we wish simply to review those results expressed in a slightly improved form.

As reported by KCS, a general solution of Eqn. (8) can be written as

$$\mathbf{\Psi}(x, \mu) = \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=3}^{4} A_\alpha \mathbf{\Psi}_\alpha(x, \mu) + \sum_{\alpha=1}^{2} \int_{-\infty}^{\infty} A_\alpha(\eta)\mathbf{F}_\alpha(\eta, \mu)\, e^{-x/\eta}\, d\eta, \tag{11}$$

where

$$\mathbf{F}_1(\mu) = \mathbf{Q}(\mu)\begin{vmatrix} 1 \\ 0 \end{vmatrix}, \qquad \mathbf{F}_2(\mu) = \mathbf{Q}(\mu)\begin{vmatrix} 0 \\ 1 \end{vmatrix}, \tag{12a}$$

$$\mathbf{\Psi}_3(x, \mu) = (\mu - x)\mathbf{F}_1(\mu) \quad \text{and} \quad \mathbf{\Psi}_4(x, \mu) = (\mu - x)\mathbf{F}_2(\mu), \tag{12b}$$

and

$$\mathbf{F}_\alpha(\eta, \mu) = \frac{1}{\sqrt{\pi}} \left[\eta \left(\frac{P}{\eta - \mu} \right) + \lambda_\alpha^*(\eta)\, \delta(\eta - \mu) \right] \mathbf{Q}(\mu)\mathbf{M}_\alpha(\eta). \tag{13}$$

Here

$$[\boldsymbol{\lambda}(\eta) - \lambda^*(\eta)\mathbf{\Psi}(\eta)]\mathbf{M}(\eta) = \mathbf{0}, \tag{14}$$

$$\boldsymbol{\lambda}(\eta) = \mathbf{I} + \eta P \int_{-\infty}^{\infty} \mathbf{\Psi}(\mu) \frac{d\mu}{\mu - \eta}, \tag{15}$$

$$\mathbf{\Psi}(\eta) = \frac{1}{\sqrt{\pi}} \tilde{\mathbf{Q}}(\eta)\mathbf{Q}(\eta)\, e^{-\eta^2}, \tag{16}$$

and the dispersion matrix is

$$\Lambda(z) = \mathbf{I} + z \int_{-\infty}^{\infty} \mathbf{\Psi}(\mu) \frac{d\mu}{\mu - z}. \tag{17}$$

If we use the continuum expansion coefficients to define a vector

$$\mathbf{A}(\eta) = A_1(\eta)\mathbf{M}_1(\eta) + A_2(\eta)\mathbf{M}_2(\eta) \tag{18}$$

and let

$$\mathbf{A}_+ = \sqrt{\pi} \begin{vmatrix} A_1 \\ A_2 \end{vmatrix} \quad \text{and} \quad \mathbf{A}_- = \sqrt{\pi} \begin{vmatrix} A_3 \\ A_4 \end{vmatrix}, \tag{19}$$

then we can express the general solution of Eqn. (8) as

$$\mathbf{\Psi}(x, \mu) = \mathbf{\Phi}(\mu)\mathbf{A}_+ + (\mu - x)\mathbf{\Phi}(\mu)\mathbf{A}_- + \int_{-\infty}^{\infty} \mathbf{\Phi}(\eta, \mu)\mathbf{A}(\eta)\, e^{-x/\eta}\, d\eta, \tag{20}$$

where

$$\mathbf{\Phi}(\mu) = \frac{1}{\sqrt{\pi}} \mathbf{Q}(\mu) \tag{21}$$

and

$$\mathbf{\Phi}(\eta, \mu) = \frac{1}{\sqrt{\pi}} \eta \left(\frac{P}{\eta - \mu} \right) \mathbf{Q}(\mu) + \delta(\eta - \mu)\, e^{\eta^2}\, \tilde{\mathbf{Q}}^{-1}(\eta)\boldsymbol{\lambda}(\eta). \tag{22}$$

In KCS a half-range expansion theorem was proved, and thus we can state here that the equation

$$\mathbf{I}(\mu) = \mathbf{\Phi}(\mu)\mathbf{A}_+ + \int_0^{\infty} \mathbf{\Phi}(\eta, \mu)\mathbf{A}(\eta)\, d\eta, \quad \mu\varepsilon(0, \infty), \tag{23}$$

has a solution for all Hölder continuous functions $\mathbf{I}(\mu)$. Also in KCS a half-range orthogonality theorem was deduced; this allows us to write

$$\int_0^{\infty} \tilde{\boldsymbol{\theta}}(\eta', \mu)\mathbf{\Phi}(\mu)\, e^{-\mu^2}\, \mu\, d\mu = \mathbf{0}, \quad \eta' > 0, \tag{24a}$$

$$\int_0^{\infty} \tilde{\boldsymbol{\theta}}(\eta', \mu)\mathbf{\Phi}(\eta, \mu)\, e^{-\mu^2}\, \mu\, d\mu = \mathbf{N}(\eta)\, \delta(\eta - \eta'), \quad \eta', \eta > 0, \tag{24b}$$

$$\int_0^{\infty} \tilde{\boldsymbol{\theta}}(\mu)\mathbf{\Phi}(\eta, \mu)\, e^{-\mu^2}\, \mu\, d\mu = \mathbf{0}, \quad \eta > 0, \tag{24c}$$

and

$$\int_0^{\infty} \tilde{\boldsymbol{\theta}}(\mu)\mathbf{\Phi}(\mu)\, e^{-\mu^2}\, \mu\, d\mu = \mathbf{N}_+. \tag{24d}$$

Here the adjoint matrices are

$$\tilde{\boldsymbol{\theta}}(\eta', \mu) = \tilde{\mathbf{\Phi}}(\eta', \mu)\tilde{\mathbf{Q}}^{-1}(\mu)\tilde{\mathbf{H}}^{-1}(\eta')\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu) \tag{25a}$$

and

$$\tilde{\boldsymbol{\theta}}(\mu) = \pi^{-1/2}\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu), \tag{25b}$$

where $\mathbf{H}(\mu)$ is the unique solution [3] of

$$\mathbf{H}(\mu) = \mathbf{I} + \mu\mathbf{H}(\mu)\int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{d\eta}{\eta+\mu}, \quad \mu\varepsilon[0, \infty), \tag{26a}$$

and

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\,d\mu = \mathbf{I}. \tag{26b}$$

In addition, the normalization vectors are given by

$$\mathbf{N}(\eta) = \pi^{-1/2}\eta[\boldsymbol{\lambda}(\eta)\boldsymbol{\Psi}^{-1}(\eta)\boldsymbol{\lambda}(\eta) + \pi^2\eta^2\boldsymbol{\Psi}(\eta)] \tag{27a}$$

and

$$\mathbf{N}_+ = \pi^{-1/2}\int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\mu\,d\mu. \tag{27b}$$

To complete our review, we note that a typical half-space problem can be solved concisely in terms of the established formalism. For example, we can write a solution of Eqn. (8) that is bounded at infinity and satisfies the free-surface condition

$$\boldsymbol{\Psi}(0, \mu) = \boldsymbol{\Psi}_{\text{inc}}(\mu), \quad \mu\varepsilon(0, \infty), \tag{28}$$

as

$$\boldsymbol{\Psi}(x, \mu) = \boldsymbol{\Phi}(\mu)\mathbf{A}_+ + \int_0^\infty \boldsymbol{\Phi}(\eta, \mu)\mathbf{A}(\eta)\,e^{-x/\eta}\,d\eta, \tag{29}$$

where

$$\mathbf{A}_+ = \mathbf{N}_+^{-1}\int_0^\infty \tilde{\boldsymbol{\theta}}(\mu)\boldsymbol{\Psi}_{\text{inc}}(\mu)\,e^{-\mu^2}\,\mu\,d\mu \tag{30a}$$

and

$$\mathbf{A}(\eta) = \mathbf{N}^{-1}(\eta)\int_0^\infty \tilde{\boldsymbol{\theta}}(\eta, \mu)\boldsymbol{\Psi}_{\text{inc}}(\mu)\,e^{-\mu^2}\,\mu\,d\mu. \tag{30b}$$

If we set $x = 0$ in Eqn. (29) and consider only negative μ, then we can write

$$\boldsymbol{\Psi}(0, -\mu) = \boldsymbol{\Phi}(-\mu)\mathbf{A}_+ + \int_0^\infty \boldsymbol{\Phi}(\eta, -\mu)\mathbf{A}(\eta)\,d\eta, \quad \mu > 0. \tag{31}$$

Upon substituting Eqns. (30) into Eqn. (31), we find that the integration over η can be performed analytically to yield the concise surface result

$$\boldsymbol{\Psi}(0, -\mu) = \int_0^\infty \mathbf{R}(\mu' \to \mu)\boldsymbol{\Psi}_{\text{inc}}(\mu')\,d\mu', \quad \mu > 0, \tag{32}$$

where

$$\mathbf{R}(\mu' \to \mu) = (\pi)^{-1/2}\frac{\mu'}{\mu'+\mu}\,\mathbf{Q}(\mu)\mathbf{H}(\mu)\tilde{\mathbf{H}}(\mu')\tilde{\mathbf{Q}}(\mu')\,e^{-\mu'^2}. \tag{33}$$

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References

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Abstract

The elementary solutions and the half-range completeness and orthogonality theorems concerning the linearized Boltzmann equation are discussed.

Zusammenfassung

Die elementaren Lösungen und die halbräumigen Vollständigkeits- und Orthogonalitätstheoreme die linearen Boltzmann-Gleichungen betreffend, werden diskutiert.

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