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The effect of anisotropic scattering on the critical slab problem in neutron transport theory using a synthetic kernel

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Abstract. The effect of widely differing scattering laws on the critical thickness of a multiplying slab in one-speed neutron transport theory is studied using a scattering kernel which consists of a linear combination of backward, forward and isotropic scattering. An extensive numerical survey is carried out for the critical thickness in order to provide benchmark results for future studies.

'Exact' solutions are obtained by two methods: the Wiener–Hopf technique and the method of elementary solutions. Although both methods lead to the same results, it is shown that the method of elementary solutions is superior, in terms of operational simplicity, for this type of problem.

1. Introduction

The importance of accurate solutions of the transport equation is well-known. For many years, the digital computer has enabled solutions of increasing degrees of complexity to be obtained in which multidimensional and multi-group effects are faithfully reproduced. Such approaches have been made through a direct numerical attack on the integro-differential or the integral form of the Boltzmann equation, e.g., by the use of S_n and collision probability techniques. However, as useful and essential as these methods can be in practical reactor analysis, it is always desirable to evaluate their accuracy by comparison with well-established bench mark problems. A particularly difficult area to test in this respect is that connected with anisotropic scattering, since even the so-called 'exact' methods have difficulties in dealing with arbitrarily anisotropic scattering. Usually only a few orders in the Legendre expansion series are manageable for exact analysis; yet, in practice, much stronger anisotropy may exist. In an attempt to overcome this problem, that is to estimate the effect of strong anisotropy on the solution, we will consider in this paper a special kernel which combines backward scattering and forward scattering with an admixture of isotropic scattering in arbitrary proportions. In this way the influence of different aspects of the scattering law may be compared. A one-speed model will be employed. In the case of purely forward scattering with an isotropic component, there are no particular problems in casting the transport equation into a suitable form for solution. On the other hand, when backward scattering is included, it is necessary to employ a transformation due to İnönü (1973, 1976) to cast the equation into canonical form.

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The purpose of this paper, therefore, is to solve exactly by analytic methods the critical problem for a slab reactor, in which fission is isotropic, but in which the scattering is a combination of completely forward, completely backward and isotropic distributions. We use two analytic techniques to solve the problem: the method of elementary solutions and the Wiener–Hopf technique (Case and Zweifel 1967, Williams 1971). This dual approach has the advantage of showing the relationship between the two methods and is therefore to be regarded as a further contribution to our understanding of transport theory.

Numerical results are obtained for the critical thickness of the slab for a variety of situations.

Lathrop (1963) has also considered the effect of extreme scattering limits, but his work was confined to a study of the infinite-medium eigenvalues of the transport equation and did not involve a boundary-value problem such as that described here.

2. The basic equations

The one-speed transport equation for the angular flux $\phi(x, \mu)$ may be written as (Case and Zweifel 1967)

$$\left(\mu \frac{\partial}{\partial x}+\Sigma\right) \phi(x, \mu)=\Sigma_s \int_{-1}^{1} \mathrm{d}\mu' f(\mu' \rightarrow \mu)\phi(x, \mu')+\frac{\bar{\nu}\Sigma_f}{2} \int_{-1}^{1} \mathrm{d}\mu' \phi(x, \mu'), \tag{1}$$

where $\Sigma=\Sigma_a+\Sigma_s$. The fission source term is taken to be isotropic in the laboratory system of coordinates, and $f(\mu' \rightarrow \mu)$ is the angular distribution of the scattered neutrons.

The cases of backward, forward and isotropic scattering can be combined by writing the scattering probability in the form

$$f(\mu' \rightarrow \mu)=l\delta(\mu+\mu')+m\delta(\mu-\mu')+\tfrac{1}{2}n \tag{2}$$

where $l+m+n=1$. Clearly l governs the proportion of backward scattering, m the proportion of forward scattering and n the proportion of isotropic scattering.

Inserting this expression into equation (1) leads to

$$\left(\mu \frac{\partial}{\partial x}+1-m\alpha_0\right) \phi(x, \mu)=l\alpha_0\phi(x, -\mu)+\tfrac{1}{2}(\beta_0+n\alpha_0) \int_{-1}^{1} \phi(x, \mu') \, \mathrm{d}\mu' \tag{3a}$$

where $\alpha_0=\Sigma_s/\Sigma$, $\beta_0=\bar{\nu}\Sigma_f/\Sigma$ and x is now in units of the total mean free path. If we scale again, such that $x(1-m\alpha_0) \rightarrow x$, we find

$$\left(\mu \frac{\partial}{\partial x}+1\right) \phi(x, \mu)=\frac{l\alpha_0}{1-m\alpha_0} \phi(x, -\mu)+\frac{1}{2} \frac{\beta_0+n\alpha_0}{1-m\alpha_0} \int_{-1}^{1} \phi(x, \mu') \, \mathrm{d}\mu'. \tag{3b}$$

Defining

$$\alpha=\frac{l\alpha_0}{1-m\alpha_0} \tag{4a}$$

and

$$\beta=\frac{\beta_0+n\alpha_0}{1-m\alpha_0} \tag{4b}$$

we obtain the final form

$$\left(\mu \frac{\partial}{\partial x}+1\right) \phi(x, \mu)=\tfrac{1}{2}\beta \int_{-1}^{1} \phi(x, \mu') \, \mathrm{d}\mu'+\alpha\phi(x, -\mu). \tag{5}$$

The character of equation (5) is seen to be very different from that normally encountered, because the arguments $\pm\mu$ cause it to be a type of integro-differential-functional equation. Before proceeding with the solution of this problem, let us note that the boundary conditions for equation (5) are

$$\phi(d, \mu)=0 \qquad (\mu<0) \tag{6a}$$

and

$$\phi(-d, \mu)=0 \qquad (\mu>0) \tag{6b}$$

where the slab is of width $2d$. We also have the symmetry condition

$$\phi(x, \mu)=\phi(-x, -\mu).$$

To convert equation (5) to canonical form, we introduce the following transformations due to İnönü (1973, 1976):

$$y=(1-\alpha^2)^{1/2}\, x \tag{7}$$

$$\phi(x, \mu) \rightarrow \Psi(y, \mu) \tag{8}$$

and

$$a=(1-\alpha^2)^{1/2}\, d. \tag{9}$$

We now find that we can express $\Psi(y, \mu)$ as

$$\Psi(y, \mu)=\frac{(b+1)}{2}\Phi(y, \mu)-\frac{(b-1)}{2}\Phi(y, -\mu) \tag{10}$$

where

$$\mu\frac{\partial}{\partial y}\Phi(y, \mu)+\Phi(y, \mu)=\tfrac{1}{2}c\int_{-1}^{1}\Phi(y, \mu')\,\mathrm{d}\mu' \tag{11}$$

with

$$\Phi(y, \mu)=\Phi(-y, -\mu) \tag{12}$$

and the boundary condition

$$\Phi(a, -\mu)=R\Phi(a, \mu) \qquad \mu>0. \tag{13}$$

Also we have defined

$$c=\frac{\beta}{1-\alpha} \tag{14}$$

$$b=\left(\frac{1-\alpha}{1+\alpha}\right)^{1/2} \tag{15}$$

and

$$R=\left(\frac{b-1}{b+1}\right) \qquad -1\leqslant R\leqslant 0. \tag{16}$$

The problem has now been reduced to that of solving equation (11), a conventional Boltzmann equation, but with the 'reflective' boundary condition given by equation (13). The complete solution can then be regained via equations (10) and (7). We shall consider the solution of equation (11) in the next section. Note, however, that when $l=0$, i.e., no backscattering, the problem reduces to the conventional one. Thus it is the presence of backscattering that places the transport equation in this anomalous form.

3. Analytical solution of the basic equation

As previously explained, two methods will be adopted for solving equation (11), i.e., the Wiener–Hopf technique and the method of elementary solutions.

3.1. Solution by the Wiener–Hopf technique

Before proceding with the solution, let us note that equation (11) subject to the boundary condition (13) can be converted to an integral equation for

$$\Phi_0(y)=\int_{-1}^{1}\Phi(y,\mu)\,\mathrm{d}\mu \tag{17}$$

which we observe from equation (10) is equal to the true total flux. Following the standard method, we find that the integral equation takes the form (Williams 1971)

$$\Phi_0(y)=\frac{c}{2}\int_{-a}^{a}\mathrm{d}y'\,\Phi_0(y')\times\left[E_1(|y-y'|)+2R\int_0^1\frac{\mathrm{d}\mu}{\mu}\,\frac{\exp(-2a/\mu)}{1-R\exp(-2a/\mu)}\cosh\left(\frac{y}{\mu}\right)\cosh\left(\frac{y'}{\mu}\right)\right]. \tag{18}$$

This equation is interesting because it shows that the effect of backscattering is related to the introduction of a partially reflecting surface at the boundaries of the slab with an albedo of R and, at the same time, contracting the true width of the slab by a factor $(1-m\alpha_0)(1-\alpha^2)^{1/2}$. In addition, we note that as $a\to\infty$: i.e., the problem becomes a Milne problem, $\Phi_0(y)$ satisfies

$$\Phi_0(y)=\frac{c}{2}\int_0^{\infty}\mathrm{d}y'\,\Phi_0(y')\,[E_1(|y-y'|)+RE_1(y+y')] \tag{19}$$

where we have shifted the left-hand slab face to $y=0$. It is important to note that this equation does not have an explicit solution in the sense of quadratures owing to the presence of the term $E_1(y+y')$ in the kernel.

Let us return now to the solution of the equation by the Wiener–Hopf technique. It is possible to apply this method directly to the integral equation, but experience shows that a more rapid and convenient approach is to apply the transform to the integro-differential equation. Thus we shift the face of the slab from $(-a, a)$ to $(0, 2a)$ and define the transform

$$\bar{\Phi}(s,\mu)=\int_0^{2a}\Phi(y,\mu)\exp(-sy)\,\mathrm{d}y. \tag{20}$$

Applying this to equation (11), we find after dividing by $(s\mu+1)$ and integrating over $\mu(-1, 1)$

$$\int_{-1}^{1}\mu\,\frac{[\Phi(0,-\mu)\exp(-2sa)-\Phi(0,\mu)]}{s\mu+1}\,\mathrm{d}\mu+\bar{\Phi}_0(s)=\frac{c}{2s}\lg\left(\frac{1+s}{1-s}\right)\bar{\Phi}_0(s) \tag{21}$$

where we have used $\Phi(2a,\mu)=\Phi(0,-\mu)$.

Inserting the boundary condition (13), viz.

$$\Phi(0,\mu)=R\Phi(0,-\mu)\qquad(\mu>0) \tag{22}$$

and rearranging, we obtain

$$[R-\exp(-2sa)]g(s)+[R\exp(-2sa)-1]g(-s)=K(s)\bar{\Phi}_0(s) \tag{23}$$

where

$$g(s)=\int_0^1 \frac{d\mu\Phi(0,\mu)}{1+s\mu} \tag{24}$$

and

$$K(s)=1-\frac{c}{2s}\lg\left(\frac{1+s}{1-s}\right). \tag{25}$$

Equation (23) is in a form suitable for Wiener–Hopf factorization. However, first we must examine the regions of analyticity of each term:

Function	Region of analyticity
$\bar{\Phi}_0(s)$	Everywhere
$K(s)$	$-1<\mathrm{Re}(s)<1$
$g(s)$	$\mathrm{Re}(s)>-1$

Since the purpose of the Wiener–Hopf technique is to arrange equation (23) so that the members on each side are analytic in overlapping half-planes, let us decompose $K(s)$ in the following way. First, define $\tau(s)$ (which is free from zeros in the strip) by

$$K(s)=\frac{s^2-\nu^2}{s^2-1}\,\tau(s) \tag{26}$$

where $\pm\nu$ are the roots of $K(s)=0$. Then we write $\tau(s)=\tau_+(s)/\tau_-(s)$, where

$$\tau_\pm(s)=\frac{1}{2\pi i}\int_{\pm\gamma-i\infty}^{\pm\gamma+i\infty}\frac{du}{u-s}\,\tau(u) \qquad \mathrm{Re}\,(\gamma)<1. \tag{27}$$

Hence $\tau_+(s)$ is analytic in $\mathrm{Re}(s)<\gamma$ and $\tau_-(s)$ is analytic in $\mathrm{Re}(s)>-\gamma$. Equation (23) is now written

$$\frac{s^2-\nu^2}{s+1}\,\frac{1}{\tau_-(s)}\,\frac{\bar{\Phi}_0(s)}{R\exp(-2sa)-1}=\frac{(s-1)}{\tau_+(s)}\left(\frac{R-\exp(-2sa)}{R\exp(-2sa)-1}\,g(s)+g(-s)\right). \tag{28}$$

Now the left-hand side of equation (28) is analytic in the half-plane $\mathrm{Re}(s)>-\gamma$, and the second term on the right-hand side is analytic in the half-plane $\mathrm{Re}(s)<\gamma$. The first term on the right-hand side is only analytic in the strip $-1<\mathrm{Re}(s)<\gamma$, and we must therefore decompose it in the following way:

$$G(s)\equiv\frac{(s-1)}{s\tau_+(s)}\,\frac{R-\exp(-2sa)}{R\exp(-2sa)-1}\,g(s)=G_+(s)-G_-(s) \tag{29}$$

where

$$G_\pm(s)=\frac{1}{2\pi i}\int_{\pm\gamma-i\infty}^{\pm\gamma+i\infty}\frac{du}{u-s}\,G(u). \tag{30}$$

We note that $G_\pm(s)$ are analytic in the half-planes $\mathrm{Re}\,(s)<\gamma$ and $>-\gamma$, respectively. Also, it was necessary to introduce the factor s into the term so that $G(s)\to 0$ as $|s|\to\infty$.

Equation (28) may now be written

$$\frac{s^2-\nu^2}{s+1}\,\frac{1}{\tau_-(s)}\,\frac{\bar{\Phi}_0(s)}{R\exp(-2sa)-1}+sG_-(s)=sG_+(s)+\frac{(s-1)}{\tau_+(s)}\,g(-s). \tag{31}$$

The left-hand side of equation (31) is analytic in the half-plane $\mathrm{Re}\,(s)>-\gamma$ and the right-hand side in the half-plane Re $(s)<\gamma$. Thus we have a strip of common analyticity which according to Liouville's theorem allows us to infer that the functions on each side of the equation are analytic continuations of each other. The behaviour at infinity shows that both sides are equal to a constant c_0. Thus

$$sG_+(s)+\frac{(s-1)}{\tau_+(s)}g(-s)=c_0. \tag{32}$$

Setting $s=0$ gives $c_0=-\tau_-(0)g(0)$, where we have noted that $\tau_-(0)=1/\tau_+(0)$. After some rearrangement, therefore

$$g(s)=\frac{1}{(s+1)\tau_-(s)}[\tau(0)g(0)-sG_+(-s)]. \tag{33}$$

The evaluation of $G_+(-s)$ is straightforward (Williams 1973), and we find

$$G_+(-s)=\frac{c}{2}\int_0^1\frac{\mathrm{d}t(1-\nu t)g(1/t)}{z(c,t)\,(1+st)H(t)}\left(\frac{R-\exp(-2a/t)}{R\exp(-2a/t)-1}\right) \tag{34}$$

where

$$z(c,t)=(1-ct\tanh^{-1}t)^2+c^2\pi^2t^2/4 \tag{35}$$

and $H(\mu)$, Chandrasekhar's H-function, is defined by (Chandrasekhar 1960)

$$H(\mu)=\frac{(1+\mu)}{(1+\nu\mu)}\tau_-\left(\frac{1}{\mu}\right). \tag{36}$$

Inserting this expression into (33), we find the following integral equation for $g(s)$:

$$g(s)=\frac{1}{(s+\nu)H(1/s)}\left[\frac{\nu}{(1-c)^{1/2}}g(0)-\frac{c}{2}s\int_0^1\frac{\mathrm{d}t(1-\nu t)\,[R-\exp(-2a/t)]}{z(c,t)(1+st)H(t)[R\exp(-2a/t)-1]}g\left(\frac{1}{t}\right)\right]. \tag{37}$$

This equation may be solved for $g(s)$ numerically. We note that for large slab thicknesses, there is no convenient 'end-point' approximation. For in the limit as $a\to\infty$, equation (37) remains implicit, and only if $R=0$ do we regain the normal, isotropic Milne problem result. Given the function $g(s)$, we are interested in three quantities: the total flux, the angular distribution at the surface, and the critical condition. The flux distribution and surface distribution are easily calculated using the left-hand side of equation (31), when we find for $R\neq0$

$$\Phi(0,-\mu)=\frac{c}{2}\frac{H(\mu)}{1-\nu\mu}\left(\tau_-(0)g(0)-\int_0^1\frac{\mathrm{d}\mu'\mu'(1+\nu\mu')H(\mu')}{\mu+\mu'}\Phi(0,-\mu')\right.$$
$$\left.\times\frac{R\exp(-2a/\mu')-1}{R-\exp(-2a/\mu')}\right)\qquad\mu>0 \tag{38}$$

and from (10) the true angular distribution is

$$\Psi(0,\mu)=\frac{2b}{1+b}\Phi(0,-\mu). \tag{39}$$

We note that for $a=\infty$, the angular flux does not reduce to the closed-form Milne problem expression unless $R=0$. This must be due to the pathological nature of back-scattering since it is unlikely that any scattering law, however anisotropic, could destroy the usual closed-form nature of the Milne solution. Similarly, we can readily show that the angular distribution is related to $\Phi_0(s)$ by

$$\Phi(0, -\mu) = \frac{1}{1 - R \exp(-2a/\mu)} \frac{c}{2\mu} \Phi_0\left(\frac{1}{\mu}\right). \tag{40}$$

The inverse transform is therefore easily applied.

As far as the critical equation is concerned, we note that because $\Phi_0(s)$ is regular everywhere we can write from equation (23)

$$[R - \exp(-2\nu a)]g(\nu) + [R \exp(-2\nu a) - 1]g(-\nu) = 0 \tag{41}$$

or

$$\exp(-2\nu a) = \frac{Rg(\nu) - g(-\nu)}{g(\nu) - Rg(-\nu)}. \tag{42}$$

Equation (42) is the critical equation and is readily shown to reduce to the conventional expression when $R = 0$.

3.2. *The method of elementary solutions*

Following the notation of Case and Zweifel (1967), we find that we can express the solution of equation (11), subject to the symmetry condition given by equation (12), as

$$\Phi(y, \mu) = A(\eta_0)\,[\phi(\eta_0, \mu) \exp(-y/\eta_0) + \phi(-\eta_0, \mu) \exp(y/\eta_0)] + \int_0^1 A(\eta)\,[\phi(\eta, \mu) \times \exp(-y/\eta) + \phi(-\eta, \mu) \exp(y/\eta)]\, d\eta \tag{43}$$

where $A(\eta_0)$ and $A(\eta)$ are the expansion coefficients to be determined by constraining the solution to satisfy equation (13). Also we note that $\eta_0 \equiv 1/\nu$ of the previous section. If we substitute equation (43) into equation (13) and let

$$D(\xi) = A(\xi)\,[\exp(a/\xi) - R \exp(-a/\xi)], \qquad \xi = \eta_0 \text{ or } \eta \in (0, 1) \tag{44}$$

we find the singular integral equation

$$D(\eta_0)\phi(\eta_0, \mu) + \int_0^1 D(\eta)\phi(\eta, \mu)\, d\eta = -D(\eta_0) \exp(-2a/\eta_0)\, W(\eta_0, a)\phi(-\eta_0, \mu) - \int_0^1 D(\eta) \exp(-2a/\eta)\, W(\eta, a)\phi(-\eta, \mu)\, d\eta \qquad \mu \in (0, 1) \tag{45}$$

where

$$W(\xi, a) = \frac{1 - R \exp(2a/\xi)}{1 - R \exp(-2a/\xi)} \qquad \xi = \eta_0 \text{ or } \eta \in (0, 1). \tag{46}$$

If we now multiply equation (45) by $\mu H(\mu)\phi(\eta_0, \mu)$, where $H(\mu)$ is Chandrasekhar's H function, and integrate over μ from 0 to 1, we find

$$D(\eta_0)\left(N(\eta_0)\, H(\eta_0) + W(\eta_0, a)\, \frac{c\eta_0}{4H(\eta_0)} \exp(-2a/\eta_0)\right) = -\int_0^1 D(\eta') \exp(-2a/\eta')\, W(\eta', a)\, \frac{c\eta'\eta_0}{2(\eta' + \eta_0)H(\eta')}\, d\eta', \tag{47}$$

where

$$N(\eta_0) = \frac{c\eta_0^3}{2}\left(\frac{c}{\eta_0^2 - 1} - \frac{1}{\eta_0^2}\right). \tag{48}$$

Similarly we can multiply equation (45) by $\mu H(\mu)\phi(\eta, \mu)$, $\eta \epsilon (0, 1)$, and integrate over μ from 0 to 1 to find

$$D(\eta)N(\eta)H(\eta) = -D(\eta_0)\exp(-2a/\eta_0)\, W(\eta_0, a)\left(\frac{c\eta\eta_0}{2(\eta+\eta_0)H(\eta_0)}\right)$$

$$-\int_0^1 D(\eta')\exp(-2a/\eta')\, W(\eta', a)\left(\frac{c\eta\eta'}{2(\eta+\eta')H(\eta')}\right) \mathrm{d}\eta' \qquad \eta \in (0, 1). \tag{49}$$

Here

$$N(\eta) = \eta\left[(1 - c\eta \tanh^{-1}\eta)^2 + \left(\frac{c\eta\pi}{2}\right)^2\right]. \tag{50}$$

It may be shown that there is a simple relationship between the $D(\eta)$ in equation (49) and $g(s)$ in equation (37).

To find the desired results for $D(\eta_0)$, $D(\eta)$ and a, we proceed to solve equations (47) and (49) iteratively. In order to initiate the iterative procedure, we take our lowest-order result to be that obtained by ignoring equation (49) entirely and by ignoring the right-hand side of equation (47); thus, without attempting to justify it rigorously, we take our lowest-order result to be $D_0(\eta) = 0$ and

$$N(\eta_0)H(\eta_0) + \frac{c\eta_0}{4H(\eta_0)}\, W(\eta_0, a_0)\exp(-2a_0/\eta_0) = 0. \tag{51}$$

Equation (51) can readily be solved to yield

$$a_0 = \frac{|\eta_0|}{2}\,\pi - z_{0,s} \tag{52}$$

where

$$z_{0,s} = \frac{\eta_0}{2}\,\lg\left(\frac{\tau - R}{1 - \tau R}\right). \tag{53}$$

Here (54)

$$\tau = \frac{4N(\eta_0)H^2(\eta_0)}{c\eta_0}$$

and since $\tau\bar{\tau} = 1$, we can readily show that $z_{0,s}$ is real. Once the critical value of a is found, then clearly equation (9) yields d. For the numerical results, we choose to report w, the critical half-width measured in units of the total mean free path. Thus in the accompanying table we report the 'exact' value of

$$w = a/[(1 - m\alpha_0)^2 - l^2\alpha_0^2]^{1/2} \tag{55}$$

where a is the result obtained by solving equations (47) and (49) iteratively. We also list w_0, the result obtained by using equation (52) in equation (55). We note that the numerical results listed in the table were established by K Neshat (1977, private communication).

4. Results and general discussion

An immediate conclusion that may be drawn from the foregoing work is that the method of elementary solutions is a very efficient technique for this type of one-speed problem. Clearly, the formalized and well-documented procedures of calculating the expansion coefficients are superior to the somewhat laborious manipulations involved in the Wiener–Hopf technique. However, these conclusions do not necessarily extend to energy-dependent or multidimensional problems where quite different considerations must be born in mind.

Our numerical results, presented in table 1, show the relative effects of different types of scattering on the critical slab thickness. In the first three $w(l, m, n)$ columns, the

Table 1. The critical half-thickness† for selected cases.

Case	α_0	β_0	$w(1, 0, 0)$	$w(0, 1, 0)$	$w(0, 0, 1)$	$w(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$w(\frac{2}{3}, \frac{1}{3}, 0)$	$w(0, \frac{2}{3}, \frac{1}{3})$	$w(\frac{1}{3}, 0, \frac{2}{3})$
1	0	1·01	8·3295	8·3295	8·3295	8·3295	8·3295	8·3295	8·3295
			(8·33)	(8·33)	(8·33)	(8·33)	(8·33)	(8·33)	(8·33)
2	0·1	1·02	1·7945	1·9259	1·8658	1·8597	1·8350	1·9052	1·8414
			(1·80)	(1·93)	(1·87)	(1·86)	(1·84)	(1·91)	(1·84)
3	0·2	1·03	1·0745	1·2134	1·1585	1·1450	1·1147	1·1944	1·1298
			(1·08)	(1·21)	(1·16)	(1·15)	(1·12)	(1·20)	(1·13)
4	0·3	1·04	0·75337	0·88717	0·84563	0·82479	0·79011	0·87356	0·81475
			(0·760)	(0·891)	(0·847)	(0·829)	(0·796)	(0·876)	(0·818)
5	0·4	1·06	0·55341	0·67549	0·65186	0·62414	0·58532	0·66968	0·62017
			(0·563)	(0·682)	(0·654)	(0·630)	(0·595)	(0·674)	(0·625)
6	0·5	1·1	0·40901	0·51532	0·51196	0·47832	0·43554	0·51891	0·48057
			(0·421)	(0·527)	(0·515)	(0·487)	(0·448)	(0·526)	(0·487)
7	0·6	1·2	0·28647	0·37062	0·38877	0·35145	0·30657	0·38429	0·35963
			(0·301)	(0·389)	(0·393)	(0·363)	(0·323)	(0·395)	(0·367)
8	0·7	1·4	0·18623	0·24715	0·28206	0·24443	0·20013	0·26858	0·25678
			(0·203)	(0·271)	(0·288)	(0·258)	(0·219)	(0·282)	(0·266)
9	0·8	1·6	0·12442	0·17295	0·21889	0·18116	0·13471	0·20051	0·19599
			(0·141)	(0·203)	(0·225)	(0·196)	(0·154)	(0·216)	(0·206)
10	0·9	1·8	0·078958	0·12118	0·17741	0·13954	0·086763	0·15595	0·15616
			(0·0925)	(0·158)	(0·184)	(0·155)	(0·104)	(0·174)	(0·166)

† The indices l, m and n in $w(l, m, n)$ correspond to the quantities given in equation (2). The results in brackets are those deduced from equation (52).

critical half-thickness is shown, respectively, for purely backward, purely forward and purely isotropic scattering. For values of β_0 close to unity we note the physically reasonable effect in which for backward scattering the critical thickness is always smaller than that for the forward or isotropic cases, i.e. leakage is reduced. On the other hand, forward scattering leads to an increased leakage and therefore a larger critical size. Isotropic scattering lies in between these two extreme limits. However, the table also shows that, for critical thicknesses less than about one mean free path, part of this physical argument fails. For example, we note that for the last four entries in the table, the critical thickness of the case with purely forward scattering is smaller than that with purely isotropic scattering. This anomalous situation can be explained by reference to the change in angular distribution which becomes highly anisotropic. However, the source of neutrons due to fission is always isotropic, and it seems possible that this more than offsets the effect of leakage. For example, we note from equation (46) that the effective c value is equal to $(\beta_0+n\alpha_0)/(1-m\alpha_0)$. For $m=0$, $n=1$, $c_I=\beta_0+\alpha_0$ and for $m=1$, $n=0$, $c_F=\beta_0/(1-\alpha_0)$. Using $\alpha_0=0{\cdot}6$ and $\beta_0=1{\cdot}2$ leads to $c_F=3$ and $c_I=1{\cdot}8$. Thus the effective multiplication constant is much greater for the forward scattering, which therefore contains a large amount of direct isotropic emission. There is, of course, a scale factor of $(1-\alpha_0)$ to be applied to the critical size to obtain true comparisons. However, the net effect for these very small slabs is clearly a gain in overall neutron economy. This curious phenomenon does not appear to have been pointed out before.

The remainder of the numerical results are composed of combinations of the three types of scattering and are given to facilitate comparison by other workers using approximate methods of solution of the transport equation. It should be added that in the case of purely forward scattering, i.e. $w(0, 1, 0)$, the scattering cross-section does not enter the problem. Also, of course we can eliminate the parameter m in favour of l and n; however, we find that a simultaneous presentation of (l, m, n) leads to greater clarity in the assessment of the importance of these three types of scattering.

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