On an exceptional case concerning plasma oscillations

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The method of elementary solutions is used to analyze the situation in plasma oscillations when there exists a real discrete eigenvalue.

I. INTRODUCTION

In two recent papers[1,2] it was argued that the original van Kampen–Case method of elementary solutions[3,4] was incomplete for a particular situation concerning the solution of the linearized Vlasov equation for a collisionless plasma. Here we show that the method of elementary solutions does, in fact, yield the correct result for the considered exceptional case. To establish the required notation, we review the development of the solutions to

$$-\frac{i}{k}\frac{\partial}{\partial t}F_k(v,t)+vF_k(v,t)+\eta(v)\int_{-\infty}^{\infty}F_k(v',t)\,dv'=0. \tag{1}$$

Here $F_k(v,t)$ is the Fourier transform of the perturbed distribution function, v is the speed, k is the transform variable, and

$$\eta(v)=-\frac{\omega_p^2}{k^2}\frac{d}{dv}f_0(v). \tag{2}$$

In Eq. (2), $f_0(v)$ represents the equilibrium distribution and ω_p is the plasma frequency,

$$\omega_p^2=4\pi Ne^2/m, \tag{3}$$

where e and m are respectively the charge and mass of the electron and N is the charge density.

On substituting solutions of the form

$$F_k(v,t)=\phi(\nu,v)\exp(-i\nu kt) \tag{4}$$

into Eq. (1), we find

$$(\nu-v)\phi(\nu,v)=\eta(v)\int_{-\infty}^{\infty}\phi(\nu,v')\,dv'. \tag{5}$$

The solutions can be normalized by taking

$$\int_{-\infty}^{\infty}\phi(\nu,v')\,dv'=1, \tag{6}$$

and thus the continuum solutions corresponding to $\nu\in(-\infty,\infty)$ can be written as

$$\phi(\nu,v)=\eta(v)\frac{P}{\nu-v}+\lambda(\nu)\delta(\nu-v), \tag{7}$$

where

$$\lambda(\nu)=1+P\int_{-\infty}^{\infty}\eta(s)\frac{ds}{s-\nu}. \tag{8}$$

If we allow ν to be complex, then the discrete solutions are

$$\phi(\nu_\alpha,v)=\frac{1}{\nu_\alpha-v}\eta(v),\qquad \alpha=1,2,3,\ldots,\kappa, \tag{9}$$

where ν_α is used to denote a zero of

$$\Lambda(z)=1+\int_{-\infty}^{\infty}\eta(v)\frac{dv}{v-z}. \tag{10}$$

If we write a general solution of Eq. (1) as

$$F_k(v,t)=\sum_{\alpha=1}^{\kappa}A_\alpha\phi(\nu_\alpha,v)\exp(-i\nu_\alpha kt)+\int_{-\infty}^{\infty}A(\nu)\phi(\nu,v)\exp(-i\nu kt)\,d\nu, \tag{11}$$

then the expansion coefficients A_α and $A(\nu)$ must satisfy the initial condition

$$F(v)=\sum_{\alpha=1}^{\kappa}A_\alpha\phi(\nu_\alpha,v)+\int_{-\infty}^{\infty}A(\nu)\phi(\nu,v)\,d\nu,\quad v\in(-\infty,\infty), \tag{12}$$

where $F(v)=F_k(v,0)$. Case has shown[4] that Eq. (12) can be solved when $F(v)$ and $A(\nu)$ are Hölder continuous functions and the discrete eigenvalues are not real. However, as Simon and Rosenbluth have pointed out,[1] there is a problem with Case's original solution for the exceptional case when any of the discrete eigenvalues becomes real and thus becomes embedded in the continuum.

In a recent elegant paper[2] Arthur, Greenberg, and Zweifel have used methods of functional analysis to develop the solution for this elusive case. We show here how the use of singular-integral equations yields the correct result.

II. ANALYSIS

For the sake of brevity, we consider that there is only one discrete eigenvalue that is embedded in the continuum and that it is a simple zero of $\Lambda(z)$. To illustrate concisely the points of principal interest here, we consider further that there are no other discrete eigenvalues. Thus we investigate

$$F(v)=A\phi(\nu_1,v)+\eta(v)P\int_{-\infty}^{\infty}A(\nu)\frac{d\nu}{\nu-v}+\lambda(v)A(v),\quad v\in(-\infty,\infty), \tag{13}$$

where $F(v)$ is an arbitrary, though specified, Hölder continuous function.[5] Here since

$$\Lambda^+(\nu_1)=\Lambda^-(\nu_1)=0, \tag{14}$$

where the + and − are used to denote limiting values as the real axis is approached from above and below, we see that

$$\lambda(\nu_1)=\eta(\nu_1)=0. \tag{15}$$

We consider $\lambda(\nu)$ and $\eta(\nu)$ to be differentiable and thus write

$$\phi(\nu_1,\nu_1)=-\eta'(\nu_1). \tag{16}$$

If we consider that $A(\nu)$ is a Hölder continuous function, then we can follow Muskhelishvili,[5] introduce

$$N(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} A(\nu) \frac{d\nu}{\nu - z}, \tag{17}$$

and thus rewrite Eq. (13) as

$$F(v) - A\phi(\nu_1, v) = N^+(v)\Lambda^+(v) - N^-(v)\Lambda^-(v), \quad v \in (-\infty, \infty), \tag{18}$$

which can be solved to yield

$$N(z) = \frac{1}{2\pi i \Lambda(z)} \int_{-\infty}^{\infty} [F(v) - A\phi(\nu_1, v)] \frac{dv}{v - z}. \tag{19}$$

However, it is clear that $N(z)$ will have a "pole" on the real axis unless we impose the conditions

$$P \int_{-\infty}^{\infty} [F(v) - A\phi(\nu_1, v)] \frac{dv}{v - \nu_1} \pm \pi i[F(\nu_1) - A\phi(\nu_1, \nu_1)] = 0, \tag{20}$$

which clearly *cannot* be satisfied for $F(v)$ arbitrary. We thus conclude that, in general, Eq. (13) has no solution with $A(\nu)$ restricted to be Hölder continuous.

Having decided that Eq. (11) does not represent a sufficiently general solution of Eq. (1), we wish to consider, for this exceptional case,

$$E_k(v, t) = B\pi i \,\mathrm{sgn}(k)\phi(\nu_1, v) \exp(-i\nu_1 kt) + \frac{B}{\nu_1 - v} [-\pi i \,\mathrm{sgn}(k)\eta(v) + \lambda(v)] \exp(-ivkt), \tag{21}$$

where B is an arbitrary constant. By direct substitution we can readily verify that Eq. (21) satisfies Eq. (1). For this special case, we now replace Eq. (11) with

$$F_k(v, t) = A\phi(\nu_1, v) \exp(-ik\nu_1 t) + \int_{-\infty}^{\infty} A(\nu)\phi(\nu, v) \exp(-ik\nu t)\, d\nu + E_k(v, t), \tag{22}$$

where $A(\nu)$ is a Hölder function. To show that the initial condition can be satisfied, we proceed to establish the solution to

$$F(v) = A\phi(\nu_1, v) + \int_{-\infty}^{\infty} A(\nu)\phi(\nu, v)\, d\nu + \frac{B}{\nu_1 - v}\lambda(v), \quad v \in (-\infty, \infty). \tag{23}$$

Again, we introduce

$$N(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} A(\nu) \frac{d\nu}{\nu - z} \tag{24}$$

and find

$$N(z) = \frac{1}{2\pi i \Lambda(z)} \int_{-\infty}^{\infty} \hat{F}(s) \frac{ds}{s - z}, \tag{25}$$

where

$$\hat{F}(s) = F(s) - A\phi(\nu_1, s) - \frac{B}{\nu_1 - s} \lambda(s). \tag{26}$$

Now to remove the "poles" from $N(z)$, we must impose the conditions

$$P \int_{-\infty}^{\infty} \hat{F}(s) \frac{ds}{s - \nu_1} = 0 \tag{27}$$

and

$$\hat{F}(\nu_1) = 0. \tag{28}$$

If we substitute Eq. (26) into Eqs. (27) and (28), then we can evaluate two of the integrals to obtain

$$P \int_{-\infty}^{\infty} F(s) \frac{ds}{\nu_1 - s} = \lambda'(\nu_1)A - \pi^2 B\eta'(\nu_1) \tag{29}$$

and

$$F(\nu_1) = -A\eta'(\nu_1) - \lambda'(\nu_1)B. \tag{30}$$

Clearly we can eliminate between Eqs. (29) and (30) to obtain

$$A = [\lambda'^2(\nu_1) + \pi^2\eta'^2(\nu_1)]^{-1} \left[\lambda'(\nu_1) \times P \int_{-\infty}^{\infty} F(s) \frac{ds}{\nu_1 - s} - \pi^2\eta'(\nu_1)F(\nu_1)\right] \tag{31}$$

and

$$B = -[\lambda'^2(\nu_1) + \pi^2\eta'^2(\nu_1)]^{-1} \left[\eta'(\nu_1) \times P \int_{-\infty}^{\infty} F(s) \frac{ds}{\nu_1 - s} + \lambda'(\nu_1)F(\nu_1)\right]. \tag{32}$$

With A and B as given by Eqs. (31) and (32), we can use Eq. (25) to find

$$A(\nu)\Lambda^+(\nu)\Lambda^-(\nu) = \int_{-\infty}^{\infty} \left[\eta(\nu) \frac{P}{\nu - s} + \lambda(\nu)\delta(\nu - s)\right] \hat{F}(s)\, ds, \tag{33}$$

and upon entering Eq. (26) into Eq. (33), we obtain

$$A(\nu) = \frac{\eta(\nu)}{\Lambda^+(\nu)\Lambda^-(\nu)} \int_{-\infty}^{\infty} [\phi^\dagger(\nu, s) - l(\nu)\phi^\dagger(\nu_1, s)] F(s)\, ds, \tag{34}$$

where

$$\phi^\dagger(\nu, s) = \frac{P}{\nu - s} + \frac{\lambda(\nu)}{\eta(\nu)} \delta(\nu - s) \tag{35}$$

and

$$l(\nu) = \frac{\Lambda^+(\nu)\Lambda^-(\nu)}{(\nu - \nu_1)\eta(\nu)} [\lambda'^2(\nu_1) + \pi^2\eta'^2(\nu_1)]^{-1}\eta'(\nu_1). \tag{36}$$

We note that $A(\nu)$ has a removable singularity at $\nu = \nu_1$.

Finally, we conclude that A as given by Eq. (31), B as given by Eq. (32), and $A(\nu)$ as given by Eq. (34) constitute the desired solution of Eq. (23).

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