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Technical Notes

Flux-Depression Factors for Scattering and Absorbing Media

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ABSTRACT

The method of elementary solutions, along with Chandrasekhar's invariance principles, is used to solve a two-region problem relevant to flux-depression calculations. An improved P-L-type approximation is also discussed.

INTRODUCTION

In a recent Note,[1] the elementary solutions[2] of the one-speed transport equation and the invariance principles of Chandrasekhar[3] were used to solve concisely and accurately the critical problem for a reflected reactor. Here we develop, in a similar manner, the solution to a two-region problem basic to flux-depression factors for, say, foil-activation or control-rod worth calculations.

We consider the one-speed transport equations for region 1, $-a \le x \le a$, and region 2, $|x| > a$, written in the familiar[4] manner

$$\mu \frac{\partial}{\partial x} \Psi_\alpha(x,\mu) + \Psi_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} \Psi_\alpha(x,\mu')d\mu' + \delta_{\alpha,2} \quad , \tag{1}$$

where $\delta_{\alpha,2}$ is used to denote the presence of a constant source throughout region 2. Here, $c_1 < 1$ and $c_2 < 1$, and thus we seek bounded solutions of Eq. (1) such that $\Psi_\alpha(-x,-\mu) = \Psi_\alpha(x,\mu)$ and $\Psi_1(a,\mu) = \Psi_2(a,\mu)$, $\mu \in (-1, 1)$. We note that Williams[5] has summarized several basic contributions to the solution of this problem and has given analytical and numerical results valid for essentially absorbing thin foils.

ANALYSIS

For region 1, we can write the angular flux as

$$\Psi_1(x,\mu) = A(\nu_0)[\phi_1(\nu_0,\mu) \exp(-x/\nu_0) + \phi_1(-\nu_0,\mu) \exp(x/\nu_0)] + \int_0^1 A(\nu)[\phi_1(\nu,\mu) \exp(-x/\nu) + \phi_1(-\nu,\mu) \exp(x/\nu)]d\nu \quad , \tag{2}$$

where we have used an established notation for Case and Zweifel's elementary solutions.[4] For region 2, we write

$$\Psi_2(x,\mu) = B(\eta_0)\phi_2(\eta_0,\mu) \exp(-x/\eta_0) + \int_0^1 B(\eta)\phi_2(\eta,\mu) \exp(-x/\eta)d\eta + \frac{1}{1 - c_2} \quad , \quad x > a \quad , \tag{3}$$

where we have added a particular solution to account for the source term. Instead of the conventional continuity condition at $x = a$, i.e., $\Psi_1(a,\mu) = \Psi_2(a,\mu)$, $\mu \in (-1, 1)$, we prefer to use Chandrasekhar's S function,

$$S_2(\mu',\mu) = \frac{c_2\mu'\mu}{\mu' + \mu} H_2(\mu')H_2(\mu) \quad , \tag{4}$$

to deduce the boundary condition

$$\Psi_1(a,-\mu) = \frac{1}{2\mu} \int_0^1 S_2(\mu',\mu)\Psi_1(a,\mu')d\mu' + \frac{1}{(1 - c_2)^{1/2}} H_2(\mu) \quad , \quad \mu \in (0, 1) \ . \tag{5}$$

Here $H_2(\mu)$ is the usual[3] H function for region 2. We note that once $\Psi_1(x,\mu)$ is constrained to satisfy Eq. (5), we can readily deduce the coefficients $B(\eta_0)$ and $B(\eta)$ required to establish $\Psi_2(x,\mu)$.

On substituting Eq. (2) into Eq. (5) and performing the indicated integration over μ', we find

$$\frac{A(\nu_0)}{H_2(\nu_0)} \exp(a/\nu_0)\phi_1(\nu_0,\mu) + \int_0^1 \frac{A(\nu)}{H_2(\nu)} \exp(a/\nu)\phi_1(\nu,\mu)d\nu = A(\nu_0)\left(\frac{c_2}{c_1} - 1\right) H_2(\nu_0) \exp(-a/\nu_0)\phi_1(-\nu_0,\mu) + \int_0^1 A(\nu)\left(\frac{c_2}{c_1} - 1\right) H_2(\nu) \exp(-a/\nu)\phi_1(-\nu,\mu)d\nu + (1 - c_2)^{-1/2} \quad , \quad \mu > 0 \quad , \tag{6}$$

which can readily be regularized, as discussed previously,[1] to yield

$$E(\nu_0) = \frac{2}{H(\nu_0)} [\exp(2z_{0,M}/\nu_0) + \exp(-2a/\nu_0)]^{-1} \times \left[\frac{2}{(c_1 - c_2)}\left(\frac{1 - c_1}{1 - c_2}\right)^{1/2} - \int_0^1 \frac{E(\nu')H(\nu')\nu'}{\nu' + \nu_0} \exp(-2a/\nu')d\nu'\right] \tag{7}$$

and

[1]C. E. SIEWERT and A. R. BURKART, *Nucl. Sci. Eng.*, **58**, 253 (1975).
[2]K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).
[3]S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, New York and London (1950).
[4]K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts (1967).
[5]M. M. R. WILLIAMS, *Proc. Phys. Soc.*, **85**, 413 (1965).

$$E(\nu)\frac{N_1(\nu)}{H(\nu)} = \nu\left(\frac{1-c_1}{1-c_2}\right)^{1/2} + \left(\frac{c_2-c_1}{2}\right) \times \nu\left[\frac{E(\nu_0)H(\nu_0)\nu_0}{\nu_0+\nu}\exp(-2a/\nu_0) + \int_0^1 \frac{E(\nu')H(\nu')\nu'}{\nu'+\nu}\exp(-2a/\nu')d\nu'\right] \quad , \quad \nu \in (0,1) \quad , \tag{8}$$

where $H(\xi) = H_2(\xi)/H_1(\xi)$, $E(\xi) = A(\xi)\exp(a/\xi)$, and $z_{0,M}$ is the two-media Milne extrapolated endpoint,[1] i.e.,

$$z_{0,M} = \frac{\nu_0}{2}\log\left[\frac{4N_1(\nu_0)}{H^2(\nu_0)(c_1-c_2)\nu_0}\right] \quad , \tag{9}$$

where

$$N_1(\nu_0) = \frac{1}{2}c_1\nu_0^3\left(\frac{c_1}{\nu_0^2-1} - \frac{1}{\nu_0^2}\right) \quad . \tag{10}$$

It is clear that Eq. (7) can be substituted into Eq. (8) to yield a Fredholm equation for $E(\nu)$; once the resulting Fredholm equation is solved numerically, $E(\nu)$ can be entered into Eq. (7) to yield $E(\nu_0)$. Clearly, once $E(\nu_0)$ and $E(\nu)$ are determined, we can find $B(\eta_0)$ and $B(\eta)$ by using either the full-range or the half-range orthogonality relations for the eigenfunctions of region 2. For example, we note that

$$B(\eta_0) = \frac{\exp(a/\eta_0)}{N_2(\eta_0)}\int_{-1}^{1}\mu\phi_2(\eta_0,\mu)\left[\Psi_1(a,\mu) - \frac{1}{1-c_2}\right]d\mu \tag{11a}$$

or

$$B(\eta_0) = \frac{\exp(a/\eta_0)}{N_2(\eta_0)H_2(\eta_0)}\int_0^1 \mu H_2(\mu)\phi_2(\eta_0,\mu)\left[\Psi_1(a,\mu) - \frac{1}{1-c_2}\right]d\mu \quad . \tag{11b}$$

EXACT RESULTS

It is clear from Eqs. (2) and (3) that we can write the desired fluxes as

$$\phi_1(x) = 2A(\nu_0)\cosh(x/\nu_0) + 2\int_0^1 A(\nu)\cosh(x/\nu)d\nu \quad , \quad x \in (-a,a) \quad , \tag{12}$$

and

$$\phi_2(x) = B(\eta_0)\exp(-x/\eta_0) + \int_0^1 B(\eta)\exp(-x/\eta)d\eta + \frac{2}{1-c_2} \quad , \quad x > a \quad . \tag{13}$$

The flux-depression factor, $\Delta = \overline{\phi}_1/\phi_2(\infty)$, can be expressed in terms of $A(\nu_0)$ and $A(\nu)$. We find

$$\Delta = \left(\frac{1-c_2}{a}\right)\left[\nu_0 A(\nu_0)\sinh(a/\nu_0) + \int_0^1 \nu A(\nu)\sinh(a/\nu)d\nu\right] \quad . \tag{14}$$

We note that the asymptotic flux in region 2 is given by

$$\phi_{2,\text{asy}}(x) = B(\eta_0)\exp(-x/\eta_0) + \frac{2}{1-c_2} \quad , \quad x > a \quad , \tag{15}$$

where $B(\eta_0)$ is given by Eq. (11a), namely,

$$B(\eta_0) = \frac{\eta_0\exp(a/\eta_0)}{N_2(\eta_0)}\left[A(\nu_0)F(\nu_0) + \int_0^1 A(\nu)F(\nu)d\nu - 1\right] \quad , \tag{16}$$

where

$$N_2(\eta_0) = \frac{1}{2}c_2\eta_0^3\left(\frac{c_2}{\eta_0^2-1} - \frac{1}{\eta_0^2}\right) \tag{17}$$

and

$$F(\xi) = \frac{\xi(c_1-c_2)}{\xi^2-\eta_0^2}\left[\xi\cosh(a/\xi) - \eta_0\sinh(a/\xi)\right] \quad . \tag{18}$$

We can now define an extrapolated endpoint z_0 and a linear extrapolation distance λ_0 by

$$\phi_{2,\text{asy}}(a - z_0) = 0 \tag{19}$$

and

$$\lambda_0 = \phi_{2,\text{asy}}(a)/\phi'_{2,\text{asy}}(a) \quad . \tag{20}$$

It follows that

$$z_0 = \eta_0\ln\left\{\frac{2N_2(\eta_0)}{\eta_0(1-c_2)\left[1 - A(\nu_0)F(\nu_0) - \int_0^1 A(\nu)F(\nu)d\nu\right]}\right\} \tag{21}$$

and

$$\lambda_0 = \eta_0[\exp(z_0/\eta_0) - 1] \quad . \tag{22}$$

In Table I we list our numerical results for selected data cases. We note that our "exact" results were obtained by using a Gaussian quadrature scheme to approximate integrals when necessary and by solving Eqs. (7) and (8) iteratively.

APPROXIMATE RESULTS

In a recent Note,[6] the principles of invariance were used with the P-L method to find an approximate value for the critical half-thickness of a reflected critical slab. Here we report the results of a similar approximate solution for the flux-depression factor.

For the source-free region 1, $-a < x < a$, we write the one-speed P-L solution as[6]

$$\Psi_1(x,\mu) = \sum_{l=0}^{L}\sum_{j=1}^{(L+1)/2}\left(\frac{2l+1}{2}\right)P_l(\mu)T_l(\nu_j)A_j \times\left[\exp(-x/\nu_j) + (-1)^l\exp(x/\nu_j)\right] \quad . \tag{23}$$

TABLE I

Numerical Results for Selected Data Cases

Data			Δ (Approximate)			Exact	
c_1	c_2	a	P-1	P-3	P-5	Δ	z_0
0.1	0.8	1.0	0.185	0.173	0.172	0.1713	0.9825
0.3	0.8	1.0	0.225	0.211	0.210	0.2090	1.0549
0.5	0.8	1.0	0.288	0.271	0.270	0.2688	1.1733
0.7	0.8	1.0	0.401	0.380	0.379	0.3780	1.4105
0.9	0.8	1.0	0.666	0.646	0.644	0.6436	2.2057
0.5	0.95	0.005	0.981	0.978	0.976	0.9695	10.6380
0.5	0.95	0.05	0.835	0.814	0.806	0.7923	5.0868
0.5	0.95	0.1	0.716	0.687	0.678	0.6678	3.7005
0.5	0.95	0.25	0.497	0.469	0.463	0.4595	2.2524
0.5	0.95	0.5	0.325	0.306	0.304	0.3027	1.5098
0.5	0.95	1.0	0.186	0.178	0.177	0.1771	1.0833
0.5	0.95	2.0	0.097	0.094	0.094	0.0938	0.9246
0.1	0.95	0.3	0.313	0.290	0.286	0.2842	1.4850
0.1	0.975	0.3	0.247	0.231	0.229	0.2277	1.5350

[6] K. NESHAT, C. E. SIEWERT, and Y. ISHIGURO, *Nucl. Sci. Eng.*, **62**, 330 (1977).

For region 2, $|x| > a$, there is a constant source, and so we write (for $x > a$)

$$\Psi_2(x,\mu) = \sum_{l=0}^{L} \sum_{j=1}^{(L+1)/2} \left(\frac{2l+1}{2}\right) P_l(\mu) T_l(\eta_j) B_j \times \exp(-x/\eta_j) + \frac{1}{1 - c_2} \quad . \tag{24}$$

Since $\Psi_1(x,\mu) = \Psi_1(-x,-\mu)$, we need only match $\Psi_1(x,\mu)$ and $\Psi_2(x,\mu)$ at $x = a$. We can now substitute Eq. (23) into Eq. (5) to obtain

$$H_2(\mu) \sum_{l=0}^{L} \sum_{j=1}^{(L+1)/2} \left(\frac{2l+1}{2}\right) \pi_l(-\mu) T_l(\nu_j) A_j [\exp(-a/\nu_j) + (-1)^l \exp(a/\nu_j)] = \frac{1}{1 - c_2} H_2(\mu) \quad , \quad \mu > 0 \quad , \tag{25}$$

where the polynomials $\pi_l(\mu)$ are those introduced previously.[6] We can multiply Eq. (25) by $P_\alpha(\mu)$, $\alpha = 1, 3, 5, \ldots, L$, integrate over μ, and solve the resulting equations for the constants A_j. In Table I, we list our results for the improved P-L calculation of the flux-depression factor Δ.

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