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THE EFFECT OF FORWARD AND BACKWARD SCATTERING ON THE LAW OF DARKENING FOR THE MILNE PROBLEM AND THE SPHERICAL ALBEDO

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Abstract—A synthetic kernel is used to study the effect of forward and backward scattering on the law of darkening for the Milne problem and the spherical albedo.

1. INTRODUCTION

In order to study the effect of forward and backward scattering, we wish to consider the equation of transfer written as

$$\mu \frac{\partial}{\partial z} \psi(z, \mu) + \rho(\sigma + \kappa)\psi(z, \mu) = \rho\sigma \int_{-1}^{1} \psi(z, \mu')f(\mu' \to \mu)\, d\mu', \tag{1}$$

where μ is the direction cosine of the propagating radiation (as measured from the positive z axis), ρ is the density of the medium in which the radiation is diffusing and σ *and* κ are respectively the scattering and absorption coefficients. Here $f(\mu' \to \mu)$ represents the scattering law, which we write as

$$f(\mu' \to \mu) = l\delta(\mu' + \mu) + m\delta(\mu' - \mu) + \frac{1}{2}n(1 + \gamma\mu\mu'). \tag{2}$$

We note that l and m are used to represent the importance of backward and forward scattering. In addition, $n = 1 - l - m$ and γ is the coefficient of linear anisotropic scattering. In terms of the optical variable

$$\tau = \rho(\sigma + \kappa)z, \tag{3}$$

we can write eqn (1) as

$$\mu \frac{\partial}{\partial \tau} \psi(\tau, \mu) + (1 - m\omega_0)\psi(\tau, \mu) = l\omega_0\psi(\tau, -\mu) + \frac{1}{2} n\omega_0 \int_{-1}^{1} \psi(\tau, \mu')(1 + \gamma\mu\mu')\, d\mu', \tag{4}$$

where

$$\omega_0 = \frac{\sigma}{\sigma + \kappa}. \tag{5}$$

We prefer to write eqn (4) in terms of the reduced optical variable

$$x = \tau(1 - m\omega_0), \tag{6}$$

and thus we consider

$$\mu \frac{\partial}{\partial x} \psi(x, \mu) + \psi(x, \mu) = \alpha\psi(x, -\mu) + \frac{1}{2}\beta \int_{-1}^{1} \psi(x, \mu')(1 + \gamma\mu\mu')\, d\mu', \tag{7}$$

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where

$$\alpha = \frac{l\omega_0}{1 - m\omega_0} \tag{8a}$$

and

$$\beta = \frac{n\omega_0}{1 - m\omega_0}. \tag{8b}$$

For the Milne problem, we seek a diverging (as $x \to \infty$) solution of eqn (7) such that

$$\lim_{x\to\infty} \psi(x, \mu)\, e^{-x} < \infty \tag{9a}$$

and

$$\psi(0, \mu) = 0, \qquad \mu > 0. \tag{9b}$$

It is clear that forward scattering does not introduce any analytical complications since the form of the resulting equation of transfer is no different from the usual equation. However, as we can see from eqn (7), backward scattering does lead to an equation of transfer considerably different from the usual one.

2. ANALYSIS

As discussed by İNÖNÜ,[1] we can reduce the problem defined by eqns (7) and (9) to a more convenient one by letting

$$y = \sqrt{(1-\alpha^2)}x \tag{10}$$

and writing

$$\psi(y, \mu) = \left(\frac{B+1}{2}\right)\Psi(y, \mu) - \left(\frac{B-1}{2}\right)\Psi(y, -\mu), \tag{11}$$

where $\Psi(y, \mu)$ is a solution to

$$\mu \frac{\partial}{\partial y}\Psi(y, \mu) + \Psi(y, \mu) = \frac{\omega}{2}\int_{-1}^{1} \Psi(y, \mu')(1 + b\mu\mu')\, d\mu'. \tag{12}$$

We note that

$$B = \sqrt{\left(\frac{1-\alpha}{1+\alpha}\right)} \tag{13}$$

and that the parameters appearing in eqn (12) are

$$\omega = \frac{\beta}{1-\alpha} \tag{14a}$$

and

$$b = \left(\frac{1-\alpha}{1+\alpha}\right)\gamma. \tag{14b}$$

The boundary condition on $\Psi(y, \mu)$ follows from substituting eqn (9b) into eqn (11):

$$\Psi(0, \mu) = R\Psi(0, -\mu), \qquad \mu > 0, \tag{15}$$

where

$$R = \frac{B-1}{B+1}. \tag{16}$$

To develop $\Psi(y,\mu)$ we can write the properly diverging (as $y\to\infty$) solution as

$$\Psi(y,\mu)=A(\nu_0)\phi(\nu_0,\mu)\,e^{-y/\nu_0}+K\phi(-\nu_0,\mu)\,e^{y/\nu_0}+\int_0^1 A(\nu)\phi(\nu,\mu)\,e^{-y/\nu}\,d\nu, \tag{17}$$

where K is a normalization constant and the elementary solutions[2] are given by

$$\phi(\pm\nu_0,\mu)=\frac{\omega\nu_0}{2}(1\pm r\nu_0\mu)\frac{1}{\nu_0\mp\mu} \tag{18a}$$

and

$$\phi(\nu,\mu)=\frac{\omega\nu}{2}(1+r\nu\mu)Pv\left(\frac{1}{\nu-\mu}\right)+\lambda(\nu)\delta(\nu-\mu). \tag{18b}$$

Here $r=b(1-\omega)$, ν_0 is the positive zero of

$$\Lambda(z)=1+\frac{\omega}{2}z\int_{-1}^{1}(1+rx^2)\frac{dx}{x-z} \tag{19a}$$

and

$$\lambda(\nu)=1+\frac{\omega}{2}\nu P\int_{-1}^{1}(1+rx^2)\frac{dx}{x-\nu}. \tag{19b}$$

If we required the complete solution, we could now substitute eqn (17) into eqn (15) and use the half-range orthogonality relations[3] to develop a Fredholm equation and an auxiliary equation that could be solved numerically to yield the expansion coefficients $A(\nu_0)$ and $A(\nu)$. However, since we seek only the law of darkening, we can use Chandrasekhar's principles of invariance[4] to write

$$\Psi(0,-\mu)=K\phi(\nu_0,\mu)+\frac{\omega}{2\mu}\int_0^1 S(\mu',\mu)[R\Psi(0,-\mu')-K\phi(\nu_0,-\mu')]\,d\mu',\ \mu>0, \tag{20}$$

where[4]

$$S(\mu',\mu)=\frac{\mu\mu'}{\mu+\mu'}H(\mu')H(\mu)[1-c(\mu+\mu')-r\mu'\mu] \tag{21}$$

with

$$c=\frac{\omega r H_1}{2-\omega H_0}. \tag{22}$$

Here the H function satisfies

$$H(\mu)=1+\frac{\omega}{2}\mu H(\mu)\int_0^1(1+rx^2)H(x)\frac{dx}{x+\mu}, \tag{23}$$

and

$$H_\alpha=\int_0^1\mu^\alpha H(\mu)\,d\mu. \tag{24}$$

We can now substitute eqn (18a) into eqn (20) and evaluate one of the integrals to obtain

$$\Psi(0,-\mu)=\frac{H(\mu)}{H(\nu_0)}K\phi\dagger(\nu_0,\mu)+\frac{\omega R}{2}H(\mu)\int_0^1\frac{xH(x)}{x+\mu}[1-c(x+\mu)-rx\mu]\Psi(0,-x)\,dx,\ \mu\in(0,1), \tag{25}$$

where

$$\phi\dagger(\nu_0,\mu)=\frac{\omega\nu_0}{2}(1+r\nu_0\mu)\frac{1}{\nu_0-\mu}+\frac{\omega\nu_0}{2}c \tag{26}$$

and

$$\frac{1}{H(\nu_0)} = 1 - \frac{\omega}{2}\nu_0 \int_0^1 (1 + rx^2)H(x)\frac{dx}{x + \nu_0}. \tag{27}$$

Equation (25) clearly is a Fredholm equation that can be solved numerically to yield $\Psi(0, -\mu)$; the law of darkening can then be obtained from eqns (11) and (15), viz.

$$\psi(0, -\mu) = \left(\frac{2B}{B+1}\right)\Psi(0, -\mu), \qquad \mu > 0. \tag{28}$$

We wish to normalize the Milne problem considered here such that

$$\int_0^1 \psi(0, -\mu)\mu \, d\mu = 1. \tag{29}$$

Equation (29) clearly determines the constant K so that eqn (25) can be written as

$$\Psi(0, -\mu) = \left(\frac{B+1}{2\nu_0 qB}\right)H(\mu)\phi^\dagger(\nu_0, \mu) + R\int_0^1 K(x \to \mu)\Psi(0, -x)dx, \mu \in (0, 1), \tag{30}$$

where

$$K(x \to \mu) = \frac{\omega}{2}\frac{H(\mu)}{\nu_0 - \mu}\left(\frac{xH(x)(\nu_0 + x)(1 + r\mu^2)}{x + \mu} - \frac{x}{q}[1 + r\nu_0\mu + c(\nu_0 - \mu)]\right). \tag{31}$$

The spherical albedo A^* is defined by AMBARTSUMYAN[5] as

$$A^* = 2\int_0^1 \psi(0, -\mu)\mu \, d\mu, \tag{32}$$

where $\psi(z, \mu)$ is a solution of eqn (1) subject to the boundary conditions

$$\psi(z, \mu) \to 0 \text{ as } z \to \infty \tag{33a}$$

and

$$\psi(0, \mu) = 1, \quad \mu > 0. \tag{33b}$$

For this problem, we can again use the transformation discussed here to find that we can express the spherical albedo as

$$A^* = -R + \frac{4B}{B+1}\int_0^1 \mu\Psi(0, -\mu)\, d\mu, \tag{34}$$

where now $\Psi(0, -\mu)$ satisfies

$$\Psi(0, -\mu) = \frac{2}{B+1}[1 - qH(\mu)] + \frac{\omega}{2}RH(\mu)\int_0^1 \frac{xH(x)}{x+\mu}[1 - c(x+\mu) - rx\mu]\Psi(0, -x)\, dx, \mu \in (0, 1), \tag{35}$$

with

$$q = \frac{2(1-\omega)}{2 - \omega H_0}. \tag{36}$$

3. NUMERICAL RESULTS

In order that our results can be used as benchmark results, we first list in Table 1 the law of darkening for $\omega_0 = 0.9$, $\gamma = 0.7$ and different values of l, m and n. In Table 2, we report the spherical albedo for selected cases.

Table 1. The law of darkening† for $\omega_0 = 0.9$ and $\gamma = 0.7$.

μ	$\psi(0,-\mu)$ $(\frac{2}{3},0,\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(0,\frac{2}{3},\frac{1}{3})$	$(\frac{1}{3},0,\frac{2}{3})$	$(0,\frac{1}{3},\frac{2}{3})$	$(0,0,1)$
0.0	0.37430	0.44138	0.58834	0.58201	0.70287	0.75057
0.1	0.53041	0.59193	0.72681	0.76427	0.87211	0.93284
0.2	0.67928	0.73461	0.85787	0.92882	1.02334	1.09176
0.3	0.83960	0.88871	1.00091	1.09871	1.17978	1.25221
0.4	1.02090	1.06351	1.16419	1.28126	1.34812	1.42019
0.5	1.23447	1.26974	1.35687	1.48290	1.53393	1.59997
0.6	1.49652	1.52268	1.59144	1.71104	1.74341	1.79565
0.7	1.83288	1.84638	1.88679	1.97528	1.98429	2.01188
0.8	2.28873	2.28233	2.27371	2.28889	2.26698	2.25430
0.9	2.95199	2.90969	2.80665	2.67129	2.60619	2.53011
1.0	4.02104	3.90182	3.59256	3.15249	3.02361	2.84887

†The indices in (l, m, n) correspond to the quantities given in eqn (2).

Table 2. The spherical albedo.†

ω_0	γ	$A^*(0,1,0)$	$A^*(0,\frac{2}{3},\frac{1}{3})$	$A^*(0,\frac{1}{3},\frac{2}{3})$	$A^*(0,0,1)$	$A^*(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$A^*(\frac{1}{3},0,\frac{2}{3})$	$A^*(\frac{2}{3},0,\frac{1}{3})$	$A^*(1,0,0)$
0.5	0.0	0.0	0.05985	0.10733	0.14654	0.15988	0.18951	0.22904	0.26795
0.7	0.0	0.0	0.12118	0.19946	0.25656	0.27183	0.31029	0.35795	0.40837
0.9	0.0	0.0	0.29528	0.40983	0.47802	0.48759	0.52966	0.57177	0.62679
0.95	0.0	0.0	0.41896	0.53391	0.59667	0.60212	0.64047	0.67440	0.72395
0.99	0.0	0.0	0.67380	0.75508	0.79456	0.79558	0.81959	0.83751	0.86761
0.999	0.0	0.0	0.88164	0.91466	0.92971	0.92978	0.93884	0.94514	0.95625
0.9	0.1	0.0	0.28971	0.40395	0.47227	0.48574	0.52694	0.57075	0.62679
0.9	0.3	0.0	0.27807	0.39156	0.46008	0.48197	0.52133	0.56868	0.62679
0.9	0.5	0.0	0.26571	0.37823	0.44688	0.47809	0.51550	0.56658	0.62679
0.9	0.7	0.0	0.25254	0.36382	0.43250	0.47412	0.50942	0.56444	0.62679
0.9	0.9	0.0	0.23845	0.34815	0.41675	0.47005	0.50307	0.56227	0.62679

†The indices in A^* (l, m, n) correspond to the quantities given in eqn (2).

We note that TEZCAN[6] has given, for $\gamma = m = 0$, some estimates of the extrapolated endpoint basic to the Milne problem.

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