On the solution of certain algebraic equations

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ABSTRACT

This note deals with the problem of determining the roots of simple algebric equations by constructing polynomial equations that have the same roots.

1. AN EXTREMA PROBLEM

The extrema problem [1] for

$$f(\xi, \eta) = |\xi|^P + |\eta|^q - 1, \; P > q > 1, \tag{1}$$

subject to the linear constraint

$$a\xi + b\eta = c, \tag{2}$$

where a, b and c are positive constants, can be reduced to seeking the roots of

$$x + \alpha x^{\beta+1} = 1, \; 0 < x < 1, \; \alpha, \beta > 0. \tag{3}$$

If β is an integer then equation (3) is a polynomial equation, and so in our discussions we will suppose that this is not the case.
To find explicit solutions, or at least to obtain polynomial equations, we first introduce the sectionally analytic function

$$\Lambda(z) = 1 - z - \alpha\, z^{\beta+1}, \tag{4}$$

where we choose the principal branch of the multivalued function $z^{\beta+1}$, so that the zeros of $\Lambda(x)$, $x > 0$, will be the desired roots of equation (3). The limiting values of $\Lambda(z)$ as z approaches the cut, i.e.,

$$\Lambda^{\pm}(t) = \lim_{\epsilon \to 0} \Lambda(t \pm i\epsilon), \; t < 0, \tag{5}$$

can be readily computed from equation (4). We find

$$\Lambda^{\pm}(t) = 1 + |t| + \alpha |t|^{\beta+1} e^{\pm i\beta\pi}, \; t < 0. \tag{6}$$

On applying the argument principle to $\Lambda(z)$ in the cut plane, we find, after writing

$$\beta = 2n + \hat{\beta}, \; -1 < \hat{\beta} < 1, \; n = 0, 1, 2, \ldots, \; \beta > 0, \tag{7}$$

that $\Lambda(z)$ has precisely 2 n + 1 zeros. We note from equation (6) that $\Lambda^{\pm}(t)$ does not vanish on the cut. It is further evident from the reflection property

$$\Lambda(z) = \overline{\Lambda(\bar{z})} \tag{8}$$

that these zeros consist of n conjugate pairs z_j and $\bar{z}_j$, j = 1, 2, ... n, and one (and only one) real positive zero x_0.
We now consider the function $X_n(z)$ defined by

$$z^{2n-\beta}\Lambda(z) = -\alpha(z - x_0) \prod_{j=1}^{n} (z - z_j)(z - \bar{z}_j) X_n(z). \tag{9}$$

Clearly $X_n(z)$ is sectionally analytic and nonzero.
We further observe on taking the limiting values of equation (9) on the cut that

$$X_n^+(t) = G_n(t)\, X_n^-(t), \; t < 0, \tag{10}$$

where

$$G_n(t) = \frac{\Lambda^+(t)}{\Lambda^-(t)} e^{-2i\pi\beta}, \; t < 0. \tag{11}$$

Now equation (10) together with the requirement that $X_n(z) \to 1$ as $|z| \to \infty$, constitutes a so-called Riemann problem [2] serving to determine $X_n(z)$ uniquely. The usual method of solution discussed by Muskhelishvili [2] leads to

$$X_n(z) = \exp\left[\frac{1}{\pi} \int_{-\infty}^{0} [\arg \Lambda^+(t) - \beta\pi] \frac{dt}{t - z}\right], \tag{12}$$

where we have chosen $\arg \Lambda^+(-\infty) = \beta\pi$. Consequently we now have the identity

$$(z - x_0) \prod_{j=1}^{n} (z - z_j)(z - \bar{z}_j) = F_n(z), \tag{13}$$

where

$$\alpha F_n(z) = -z^{2n-\beta} \Lambda(z)\, X_n^{-1}(z). \tag{14}$$

Of interest in this particular example is the real root x_0. An explicit expression can, of course, be given for the n = 0 case. On setting z = 1, for instance, we have

$$x_0 = 1 - X_0^{-1}(1), \; 0 < \beta < 1, \tag{15}$$

where from equation (12)

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$$X_0^{-1}(1) = \exp\left[\frac{1}{\pi}\int_0^{\infty}\tan^{-1}\left(\frac{-(1+t)\sin\beta\pi}{(1+t)\cos\beta\pi + at^{\beta+1}}\right)\frac{dt}{t+1}\right],$$
$$0 < \beta < 1. \qquad (16)$$

The determination of x_o is thus reduced to quadrature. The $n = 1$ case can be treated in a similar manner.

As an alternative approach to seeking x_o for $0 < \beta < 1$, we can use the theory of residues [3] to write

$$\frac{1}{2\pi i}\int\frac{\Lambda'(z')}{\Lambda(z')}\frac{dz'}{z'-z} = \frac{\Lambda'(z)}{\Lambda(z)} + \frac{1}{x_0 - z}, \quad 0 < \beta < 1, \qquad (17)$$

where the contour integral consists of a path around the cut, that we allow to shrink onto the cut, joined to a circle, centered at the origin, whose radius we allow to tend to infinity. Since the contribution to the integral tends to zero as we allow the radius of the circle to increase without bound, we can write equation (17) as

$$\frac{1}{x_0 - z} = -\frac{\Lambda'(z)}{\Lambda(z)} + \frac{1}{2\pi i}\int_{-\infty}^{0}[F^+(t) - F^-(t)]\frac{dt}{t-z},$$
$$0 < \beta < 1, \qquad (18)$$

where $F(z) = \Lambda'(z)/\Lambda(z)$. We can now use equation (6) to write equation (18) as

$$\frac{1}{x_0 - z} = -\frac{\Lambda'(z)}{\Lambda(z)}$$
$$+\frac{a}{\pi}\sin\beta\pi\int_0^{\infty}\frac{t^{\beta}(1+\beta+\beta t)}{(1+t)^2 + 2a(1+t)\,t^{\beta+1}\cos\beta\pi + (at^{\beta+1})^2}\frac{dt}{t+z},$$
$$0 < \beta < 1. \qquad (19)$$

Equation (19) can of course be solved for x_0 to yield

$$x_0 = z + \Big[-\frac{\Lambda'(z)}{\Lambda(z)}$$
$$+\frac{a}{\pi}\sin\beta\pi\int_0^{\infty}\frac{t^{\beta}(1+\beta+\beta t)}{(1+t)^2 + 2a(1+t)\,t^{\beta+1}\cos\beta\pi + (at^{\beta+1})^2}\frac{dt}{t+z}\Big]^{-1},$$
$$0 < \beta < 1, \qquad (20)$$

where z is an arbitrary parameter. From a numerical point of view, we have found that on using

$$z = z_0 = \frac{a - 1 + (a+1)(1+4a)^{1/2}}{4a(a+1)} \qquad (21)$$

in equation (20) results correct to five significant figures could be obtained.

2. THE GEOMETRIC SERIES

For a typical annuity calculation we consider [4] the geometric series

$$S = R(1 + V + V^2 + \ldots + V^{m-1}), \qquad (22)$$

where R is the periodic payment or rent, S is the amount of ordinary annuity of m payments and $V = (1 + I)^{-1}$, with I being the interest rate per payment period. Equation (22) clearly can be summed to yield

$$S = \frac{R(V^m - 1)}{(V - 1)}. \qquad (23)$$

For a given m and $K = S/R$, $1 < K < m$, we wish to solve equation (23) for the required interest rate. If we let

$$V = [(K-1)\,x]^{1/m} \qquad (24)$$

and

$$a = K\,(K-1)^{\frac{1-m}{m}} \qquad (25)$$

we can rewrite equation (23) as

$$1 = -x + a\,x^{1+\beta} \qquad (26)$$

where

$$\beta = \frac{1-m}{m} \qquad (27)$$

Equation (26) clearly is similar to equation (3) and thus we let

$$\Omega(z) = 1 + z - az^{\beta+1} \qquad (28)$$

and note that $\Omega(z)$ has two zeros (real) in the plane cut along the negative real axis. On considering the Riemann problem analogous to that defined by equation (10), we find we can write

$$\Omega(z) = (z - x_1)(z - x_0)\,X(z), \qquad (29)$$

where

$$X(z) = \frac{1}{z}\exp\left[-\frac{1}{\pi}\int_0^{\infty}\tan^{-1}\left(\frac{-at^{\beta+1}\sin\beta\pi}{-1 + t - at^{\beta+1}\cos\beta\pi}\right)\frac{dt}{t+z}\right] \qquad (30)$$

and

$$x_1 = \frac{1}{K-1}. \qquad (31)$$

We can now solve equation (29) for x_0:

$$x_0 = z - \frac{\Omega(z)\,X^{-1}(z)}{z - x_1}. \qquad (32)$$

We have found that

$$z = z_0 = \frac{1}{K-1}\left(\frac{2K-1}{2K}\right)^m \qquad (33)$$

can be used in equation (32) to yield results accurate to five significant figures.

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