Temperature-jump problem with arbitrary accommodation

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A concise and accurate result for the temperature-jump coefficient based on the linearized BGK model and arbitrary accommodation is reported. The jump coefficient is expressed as a power series in $(1-\alpha)$, and values of the expansion coefficients are given.

We wish to give a rearranged version of some previous results for the temperature-jump coefficient. As discussed earlier, the temperature-jump coefficient can be expressed as

$$\epsilon' = \left(2 - \frac{\alpha}{2}\right) \frac{5}{\sqrt{\pi}} \epsilon_0 + \Delta(\alpha),$$

where $\alpha$ is the accommodation coefficient for the linearized BGK model,

$$\epsilon_0 = \frac{8}{5\sqrt{\pi}} \left| \begin{array}{c|c} 1 & 1 \\ \hline 0 & -1 \end{array} \right|,$$

and

$$\Delta(\alpha) = \left(\frac{1 - \alpha}{\alpha\sqrt{\pi}}\right) \left| \begin{array}{c|c} 1 & 0 \\ \hline 0 & \sqrt{\frac{\pi}{2}} \int_0^\infty H_{-1}^t \left( \frac{\pi}{\sqrt{\eta}} \right) d\eta \end{array} \right| \times A(\eta) \eta \exp(-\eta^2) d\eta.$$

Here, $H(\mu)$ is the $2 \times 2$ matrix introduced by Kriese et al.,

$$H(t) = \left( \frac{\pi}{\sqrt{2}} \right) \int_0^\infty H_t(\mu) Q(\mu) \mu^t \exp(-\mu^2) d\mu,$$

where

$$Q(\mu) = \left[ \begin{array}{c} \frac{\mu_3}{\sqrt{3}} - \frac{\mu_2}{\sqrt{3}} \\ \frac{\mu_3}{\sqrt{3}} \end{array} \right],$$

and the vector $A(\eta)$ is the solution of the Fredholm equation

$$A(\eta) = (\alpha - 2) F(\eta) + (1 - \alpha) (TA)(\eta), \quad \eta > 0.$$  

In addition, we use the superscript $t$ to denote the transpose operation and the superscript $-t$ to denote the transpose-inverse operation. In Eq. (6)

$$F(\eta) = \eta \exp(-\eta^2) R(\eta) H_{-1}^t(\eta) H_t^t,$$

where

$$R(\eta) = \frac{1}{N(\eta)} \left| \begin{array}{c|c} \frac{\sqrt{2}}{\sqrt{\pi}} N_{12}(\eta) & -N_{12}(\eta) \\ \hline \sqrt{\frac{2}{\pi}} N_{12}(\eta) & N_{11}(\eta) \end{array} \right|,$$

We note that Kriese et al. have given $N_{12}(\eta)$ and $N(\eta)$ in terms of Dawson's integral. The operator $T$ in Eq. (6) is such that

$$(T X)(\eta) = \frac{\eta \exp(-\eta^2)}{\sqrt{\pi}} R(\eta) H_{-1}^t(\eta) \times \int_0^\infty H_{-1}^t(\mu) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] X(\mu) \exp(-\mu^2) \mu \frac{d\mu}{\mu + \eta}.$$  

We have computed the Hilbert–Schmidt norm of the kernel $K$ of the operator $T$, where

$$K(\eta, \mu) = \frac{\eta \exp(-\eta^2)}{\sqrt{\pi}} R(\eta) H_{-1}^t(\eta) \times H_{-1}^t(\mu) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \frac{\mu \exp(-\mu^2)}{\mu + \eta}.$$  

We used the definition

$$|| T || = \left\{ \int_0^\infty \int_0^\infty d\eta \int_0^\infty d\mu \text{tr}[K(\eta, \mu)K(\mu, \eta)] \right\}^{1/2},$$

where

$$\text{tr}[K(\eta, \mu)] = K_{11}(\eta, \mu) + K_{12}(\eta, \mu) + K_{21}(\eta, \mu) + K_{22}(\eta, \mu).$$

We find that $|| T || \sim 0.04$; that is, for $|1 - \alpha| \lesssim 0.04$, or $\alpha \lesssim 25$, it should be possible to obtain a convergent Neumann--Liouville iterative solution of Eq. (6). In fact for physically interesting cases $\alpha \lesssim 1$ and in this range, one can expect a very rapidly convergent iterative solution.
Thus, we now express an iterative solution of Eq. (6) as

$$A(\eta) = (\alpha - 2) \left[ F(\eta) + \sum_{m=1}^{n} (1 - \alpha)^{m} (T^{m}F)(\eta) \right];$$

(13)

then, we can write the jump coefficient as

$$\epsilon' = \frac{5 \sqrt{\pi}}{8} \left( \frac{2 - \alpha}{\alpha} \right) \left[ \epsilon_{0} + \sum_{m=1}^{n} (1 - \alpha)^{m} \epsilon_{m} \right],$$

(14)

where

$$\epsilon_{m} = \frac{1}{\sqrt{\pi}} \frac{8}{5 \sqrt{\pi}} \left[ \begin{array}{c} 1 \\ 0 \\ \frac{1}{\sqrt{\pi}} \end{array} \right] H_{m-1}^{t} H_{m}^{t} \left[ \begin{array}{c} \sqrt{\eta} \\ -1 \\ 0 \end{array} \right], \quad m > 0,$$

(15)

$$W_{m} = \int_{0}^{\infty} H^{t}(\eta) \left[ \begin{array}{c} 1 \\ 0 \\ \frac{\sqrt{\eta}}{\sqrt{\pi}} \end{array} \right] (T^{m}B)(\eta) \eta \exp(-\eta^{2}) d\eta,$$

(16)

and

$$B(\eta) = \eta \exp(-\eta^{2}) R(\eta) H^{t}(\eta).$$

(17)

Equation (11) is a convenient result because the $\epsilon_{m}$ do not depend on $\alpha$. We have evaluated Eqs. (2) and (15) numerically to find the results given in Table I. A Gaussian quadrature scheme was used to evaluate all integrals, and the number of quadrature points was increased until no change in Table I was observed. The final calculation utilized 80 Gauss-Legendre points in the interval $(0,1)$, which was mapped onto $(0, \infty)$ according to the transformation $\mu = \mu'(1-\mu^{2})$. The accuracy of the present results to the significant digits reported here is verified by the fact that Eqs. (15) give

$$\sum_{m=2}^{12} \epsilon_{m} = 1.000000;$$

that is, in the limit $\alpha \to 0$, from Eq. (11) we get

$$\epsilon' = \frac{5 \sqrt{\pi}}{8} \left( \frac{2 - \alpha}{\alpha} \right).$$

(18)

which is an exact result in this limit. Since in view of the expansion in $(1 - \alpha)$ one would expect the maximum numerical error at $\alpha = 0$, and since in this limit our result is exact to the number of significant digits quoted, it is clear that the present series expansion should provide exact values of $\epsilon'$ to the number of significant digits quoted here.

Table I. Numerical values for $\epsilon_{m}$ as computed from Eq. (15).

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<th>$m$</th>
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<td>1</td>
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</tr>
</tbody>
</table>

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\footnote{J. R. Thomas, Jr., Phys. Fluids 16, 1162 (1973).}

