# Temperature-jump problem with arbitrary accommodation

S. K. Loyalka

*Nuclear Engineering Department, University of Missouri-Columbia, Columbia, Missouri 65201*

C. E. Siewert[a)]

*Laboratoire d'Optique Atmospherique, Université des Sciences et Techniques de Lille, France*

J. R. Thomas, Jr.

*Nuclear Engineering Group, Mechanical Engineering Department, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061*

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A concise and accurate result for the temperature-jump coefficient based on the linearized BGK model and arbitrary accommodation is reported. The jump coefficient is expressed as a power series in (1-$\alpha$), and values of the expansion coefficients are given.

We wish to give a rearranged version of some previous results[1] for the temperature-jump coefficient. As discussed earlier[1] the temperature-jump coefficient can be expressed as

$$\epsilon' = \left(\frac{2-\alpha}{\alpha}\right)\frac{5\sqrt{\pi}}{8}\epsilon_0 + \Delta(\alpha), \tag{1}$$

where $\alpha$ is the accommodation coefficient for the linearized BGK model,

$$\epsilon_0 = \frac{8}{5\sqrt{\pi}} \begin{vmatrix} 1 \\ 0 \end{vmatrix}^t \mathbf{H}_1^{-t}\mathbf{H}_2^t \begin{vmatrix} 1 \\ -\sqrt{\frac{2}{3}} \end{vmatrix}, \tag{2}$$

and

$$\Delta(\alpha) = \frac{(1-\alpha)}{\alpha\sqrt{\pi}} \begin{vmatrix} 1 \\ 0 \end{vmatrix}^t \mathbf{H}_1^{-t}\int_0^\infty \mathbf{H}^{-1}(\eta) \begin{vmatrix} 1 & 0 \\ 0 & \sqrt{\frac{2}{3}} \end{vmatrix} \times \mathbf{A}(\eta)\,\eta\exp(-\eta^2)\,d\eta. \tag{3}$$

Here, $\mathbf{H}(\mu)$ is the $2\times 2$ $\mathbf{H}$ matrix introduced by Kriese *et al.*,[2]

$$\mathbf{H}_\beta^t = (\pi)^{-1/2}\int_0^\infty \mathbf{H}^t(\mu)\mathbf{Q}^t(\mu)\mathbf{Q}(\mu)\,\mu^\beta\exp(-\mu^2)\,d\mu, \tag{4}$$

$$\mathbf{Q}(\mu) = \begin{vmatrix} \sqrt{\frac{2}{3}}(\mu^2-\frac{1}{2}) & 1 \\ \sqrt{\frac{2}{3}} & 0 \end{vmatrix}, \tag{5}$$

and the vector $\mathbf{A}(\eta)$ is the solution of the Fredholm equation

$$\mathbf{A}(\eta) = (\alpha-2)\mathbf{F}(\eta) + (1-\alpha)(\mathbf{TA})(\eta), \quad \eta > 0. \tag{6}$$

In addition, we use the superscript $t$ to denote the transpose operation and the superscript $-t$ to denote the transpose-inverse operation. In Eq. (6)

$$\mathbf{F}(\eta) = \eta\exp(-\eta^2)\mathbf{R}(\eta)\mathbf{H}^{-t}(\eta)\mathbf{H}_1^t \begin{vmatrix} \sqrt{\frac{3}{2}} \\ -1 \end{vmatrix}, \tag{7}$$

where

$$\mathbf{R}(\eta) = \frac{1}{N(\eta)} \begin{vmatrix} \sqrt{\frac{3}{2}}N_{22}(\eta) & -N_{12}(\eta) \\ -\sqrt{\frac{3}{2}}N_{12}(\eta) & N_{11}(\eta) \end{vmatrix}. \tag{8}$$

We note that Kriese *et al.*[2] have given $N_{ij}(\eta)$ and $N(\eta)$ in terms of Dawson's integral. The operator $\mathbf{T}$ in Eq. (6) is such that

$$(\mathbf{TX})(\eta) = \frac{\eta\exp(-\eta^2)}{\sqrt{\pi}}\mathbf{R}(\eta)\mathbf{H}^{-t}(\eta) \times\int_0^\infty \mathbf{H}^{-1}(\mu) \begin{vmatrix} \sqrt{\frac{3}{2}} & 0 \\ 0 & 1 \end{vmatrix} \mathbf{X}(\mu)\exp(-\mu^2)\,\mu\,\frac{d\mu}{\mu+\eta}. \tag{9}$$

We have computed the Hilbert–Schmidt[3] norm of the kernel $\mathbf{K}$ of the operator $\mathbf{T}$, where

$$\mathbf{K}(\eta,\mu) = \frac{\eta\exp(-\eta^2)}{\sqrt{\pi}}\mathbf{R}(\eta)\mathbf{H}^{-t}(\eta) \times\mathbf{H}^{-1}(\mu) \begin{vmatrix} \sqrt{\frac{3}{2}} & 0 \\ 0 & 1 \end{vmatrix} \frac{\mu\exp(-\mu^2)}{\eta+\mu}. \tag{10}$$

We used the definition

$$||\mathbf{T}|| = \left\{\int_0^\infty d\eta\int_0^\infty d\mu\,\mathrm{tr}[\mathbf{K}^t(\mu,\eta)\mathbf{K}(\mu,\eta)]\right\}^{1/2}, \tag{11}$$

where

$$\mathrm{tr}[\mathbf{K}^t\mathbf{K}] = K_{11}^2 + K_{12}^2 + K_{21}^2 + K_{22}^2. \tag{12}$$

We find that $||\mathbf{T}|| \simeq 0.04$; that is, for

$$|(1-\alpha)| \lesssim (0.04)^{-1},$$

or $\alpha \lesssim 25$, it should be possible to obtain a convergent Neumann–Liouville iterative solution of Eq. (6). In fact for physically interesting cases $\alpha \le 1$ and in this range, one can expect a very rapidly convergent iterative solution.

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Thus, we now express an iterative solution of Eq. (6) as

$$\mathbf{A}(\eta)=(\alpha-2)\left[\mathbf{F}(\eta)+\sum_{m=1}^{\infty}(1-\alpha)^m(\mathbf{T}^m\mathbf{F})(\eta)\right]; \tag{13}$$

then, we can write the jump coefficient as

$$\epsilon'=\frac{5\sqrt{\pi}}{8}\left(\frac{2-\alpha}{\alpha}\right)\left[\epsilon_0+\sum_{m=1}^{\infty}(1-\alpha)^m\epsilon_m\right], \tag{14}$$

where

$$\epsilon_m=-\frac{1}{\sqrt{\pi}}\frac{8}{5\sqrt{\pi}}\begin{vmatrix}1\\0\end{vmatrix}^t\mathbf{H}^{-t}\mathbf{W}_{m-1}\mathbf{H}_1^t\begin{vmatrix}\sqrt{\frac{3}{2}}\\-1\end{vmatrix},\quad m>0, \tag{15}$$

$$\mathbf{W}_m=\int_0^{\infty}\mathbf{H}^{-1}(\eta)\begin{vmatrix}1 & 0\\0 & \sqrt{\frac{2}{3}}\end{vmatrix}(\mathbf{T}^m\mathbf{B})(\eta)\,\eta\exp(-\eta^2)\,d\eta, \tag{16}$$

and

$$\mathbf{B}(\eta)=\eta\exp(-\eta^2)\mathbf{R}(\eta)\mathbf{H}^{-t}(\eta). \tag{17}$$

Equation (11) is a convenient result because the $\epsilon_m$ do not depend on $\alpha$. We have evaluated Eqs. (2) and (15) numerically to find the results given in Table I. A Gaussian quadrature scheme was used to evaluate all integrals, and the number of quadrature points was increased until no change in Table I was observed. The final calculation utilized 80 Gauss-Legendre points[4] in the interval (0, 1), which was mapped onto $(0,\infty)$ according to the transformation $\mu=\mu'/(1-\mu')$. The accuracy of the present results to the significant digits reported here is verified by the fact that Eqs. (15) give,

$$\sum_{i=0}^{10}\epsilon_i=1.000000;$$

that is, in the limit $\alpha\to 0$, from Eq. (11) we get

$$\epsilon'=\frac{5\sqrt{\pi}}{8}\left(\frac{2-\alpha}{\alpha}\right),$$

TABLE I. Numerical values for $\epsilon_m$ as computed from Eq. (15).

| $m$ | $\epsilon_m$ |
|---|---|
| 0 | 1.17597 |
| 1 | $-1.60683\times10^{-1}$ |
| 2 | $-1.37349\times10^{-2}$ |
| 3 | $-1.38665\times10^{-3}$ |
| 4 | $-1.44586\times10^{-4}$ |
| 5 | $-1.52085\times10^{-5}$ |
| 6 | $-1.60481\times10^{-6}$ |
| 7 | $-1.69582\times10^{-7}$ |
| 8 | $-1.79329\times10^{-8}$ |
| 9 | $-1.89710\times10^{-9}$ |
| 10 | $-2.00735\times10^{-10}$ |

which is an exact result in this limit. Since in view of the expansion in $(1-\alpha)$ one would expect the maximum numerical error at $\alpha\to 0$, and since in this limit our result is exact to the number of significant digits quoted, it is clear that the present series expansion should provide exact values of $\epsilon'$ to the number of significant digits quoted here.

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[a]Permanent address: Nuclear Engineering Department, North Carolina State University, Raleigh, N.C. 27607.

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