On a possible experiment to evaluate the validity of the one-speed or constant cross section model of the neutron-transport equation

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(Received 16 January 1978)

The inverse problem for a half-space is solved (for isotropic scattering) to yield results that suggest an idealized experiment that could be used to evaluate in a new way the validity of the one-speed or constant cross section model of the neutron-transport equation.

INTRODUCTION

Inverse problems in the theory of neutron diffusion have been discussed in recent years for finite and infinite media. Here we would like to investigate the half-space inverse problem for the one-speed or constant cross section model of the neutron-transport equation and to show how the established results suggest an experiment that could be used to evaluate the isotropic-scattering model of the neutron-transport equation.

ANALYSIS

We consider the neutron-transport equation

\[
\frac{2}{\Sigma c} \varphi(x, \mu) + \varphi(x, \mu) = \frac{c}{2} \int_{0}^{1} \varphi(x, \mu') d\mu',
\]

where \(\varphi(x, \mu)\) is the neutron angular flux, \(x\) is the position variable measured in mean-free-paths, \(\mu\) is the direction cosine, and

\[
c = (\sigma_{s} + \Sigma_{a}) / \Sigma
\]

is the mean number of secondary neutrons per collision. Traditionally for \(c < 1\), we seek to solve Eq. (1) in a semi-infinite half-space such that

\[
\varphi(0, \mu) = F(\mu), \quad \mu > 0
\]

and

\[
\varphi(\infty, \mu) = 0,
\]

where \(F(\mu)\) is considered given. Here we consider that \(F(\mu)\) is specified, that \(\varphi(0, -\mu), \mu > 0\), can be measured experimentally, and that we wish to determine the mean number of secondaries \(c\).

We know from the work of Chandrasekhar that the exit flux can be computed from

\[
\varphi(0, -\mu) = \frac{c}{2} H(\mu) \int_{0}^{1} H(x) F(x) x \frac{dx}{x + \mu}, \quad \mu > 0,
\]

where \(H(\mu)\) satisfies

\[
H(\mu) = 1 + \frac{c}{2} \varphi H(\mu) \int_{0}^{1} H(x) x \frac{dx}{x + \mu},
\]

It is clear that we cannot readily solve Eq. (4) for \(c\). Since \(H(\mu)\) is a function of \(c\), Moments of the exit distribution can be found by multiplying Eq. (4) by \(\mu^{a}\) and integrating over \(\mu\). For example, after using Eq. (5), we can write

\[
\psi_{0} = \frac{c}{2} \mu \int_{0}^{1} F(x) \left[ x H(x) \left( \frac{c}{2} H_{1} + x \sqrt{1 - c} \right) \right] dx,
\]

and

\[
\psi_{1} = \frac{c}{2} \mu \int_{0}^{1} F(x) \left[ x H(x) \left( \frac{c}{2} H_{1} + x \sqrt{1 - c} \right) \right] dx,
\]

where

\[
H_{0} = \frac{c}{2} \mu \int_{0}^{1} H(x) x \mu^{a} dx
\]

and

\[
\psi_{a} = \frac{c}{2} \mu \int_{0}^{1} \varphi(0, -\mu) \mu^{a} d\mu,
\]

If we consider the special case of an isotropic incident flux, \(F(\mu) = 1\), then the resulting version of Eq. (6a) yields

\[
\psi_{0}^{(0)} = \psi_{1}^{(0)} = \psi_{2}^{(0)} = H_{0} - 1 = (2/c)(1 - \sqrt{1 - c}) - 1,
\]

which can be solved for \(c\) to yield

\[
c = \frac{4 \psi_{0}^{(1)}}{\psi_{0}^{(0)} + 1}.
\]

Here we use

\[
\psi_{1}^{(0)} = \int_{0}^{1} \varphi(0, -\mu) \mu^{a} d\mu,
\]

where \(\varphi^{(a)}(x, \mu)\) denotes the solution of Eq. (1) corresponding to \(F(\mu) = \mu^{a}\).

If we now consider \(F(\mu) = \mu\), then Eqs. (6a) and (6b) can be used with the identity

\[
\sqrt{1 - c} H_{1} + (c/4) H_{1}^{2} = \frac{1}{2}
\]

to deduce

\[
c = \frac{4 \psi_{1}^{(1)}}{\psi_{1}^{(0)} + 1}.
\]

In a similar manner Eqs. (6a) and (6c) and the identity

\[
\sqrt{1 - c} H_{1} + (c/2)(\frac{1}{2} H_{1}^{2} - H_{1} H_{1}) = \frac{1}{2}
\]

can be used to establish

\[
c = \frac{4 \psi_{2}^{(0)}}{\psi_{2}^{(0)} + 1}.
\]
With the aid of Busbridge's identity\textsuperscript{7} concerning moments of the $H$ function,
\[
\sqrt{1 - cH_{2\alpha}} + \frac{c}{4}(H_{1}H_{2\alpha-1} - H_{2}H_{2\alpha-2} + \cdots + H_{2\alpha-\beta}H_{\beta})
\]
\[
= \frac{1}{2c + 1},
\]
we can generalize Eqs. (9), (12), and (14) to obtain
\[
c = \frac{4\phi^{(\beta)}_{0}}{\phi^{(\beta)}_{0} + (\beta + 1)^{-1}P}, \quad \beta = 0, 1, 2, 3, \cdots.
\]

Generally when we apply Eq. (1) to physical problems we consider $c$ to be a constant and thus clearly not a function of the boundary conditions. It thus seems feasible that the manner in which $c$, as computed from Eq. (16) and the experimentally measured $\phi^{(\beta)}(0, -\hat{\mu})$, varies with $\beta$ would be a reasonable measure of the accuracy with which Eq. (1) represents the given physical problem. It also seems feasible that the multigroup version of Eq. (16) would offer a definition of the transfer cross sections alternative to the traditional one. The finite-slab inverse problem solved in Ref. 2 for the multigroup model could serve a similar purpose.

\textbf{ACKNOWLEDGMENTS}

The author is grateful to E. İnönü and Boğaziçi Üniversitesi for their kind hospitality and partial support of this work and to T. Yarman whose queries concerning the inverse problem were partly responsible for some of the work reported here. This work was also partially supported by the Centre d'Etudes Nucléaires de Saclay and the National Science Foundation through grant ENG. 7709405.

\textsuperscript{1}S. Phor, Phys. Rev. 175, 218 (1968).