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Technical Notes

On Several Good Methods for Computing the Distortion Factor Relevant to a Foil Placed in an Exponentially Varying Flux

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ABSTRACT

The method of elementary solutions, the C_N method, and the integral transform method are used to compute the flux-distortion factor for a foil placed in an exponentially varying flux.

I. INTRODUCTION

In two recent papers,[1,2] the earlier work of Williams[3] on flux-depression factors due to a constant source was extended to allow the foil to scatter as well as absorb neutrons. Here we wish to give the results for a similar extension when we consider a foil placed in an exponentially varying flux.[4]

We consider the foil to be region 1, $x \in (-a,a)$, and that in the foil and region 2, $|x| > a$, the neutron angular flux is defined by the one-speed neutron transport equation,

$$\mu \frac{\partial}{\partial x} F_\alpha(x,\mu) + F_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} F_\alpha(x,\mu')d\mu' \quad ,$$

$$\alpha = 1 \text{ and } 2 \quad . \tag{1}$$

Here we seek, for $c_1 < 1$ and $c_2 < 1$, solutions of Eq. (1) such that

$$F_1(\pm a,\mu) = F_2(\pm a,\mu) \quad , \quad \mu \in (-1,1) \quad , \tag{2a}$$

$$\int_{-1}^{1} F_2(x,\mu)d\mu \to \exp(x/\eta_0) \quad , \quad \text{as} \quad x \to \infty \quad , \tag{2b}$$

and

$$\int_{-1}^{1} F_2(-x,\mu)d\mu \to 0, \quad \text{as} \quad x \to \infty \quad , \tag{2c}$$

where η_0 is the discrete eigenvalue for region 2. Explicitly, we wish to compute the flux-distortion factor,

$$\Delta = \frac{\dfrac{1}{2a} \displaystyle\int_{-a}^{a} \int_{-1}^{1} F_1(x,\mu)d\mu dx}{\dfrac{1}{2a} \displaystyle\int_{-a}^{a} \exp(x/\eta_0)\, dx} \quad . \tag{3}$$

From Eq. (3), it is clear that Δ is normalized so that $\Delta = 1$ when $c_1 = c_2$, i.e., when there is no foil. Before discussing the solution, we find it convenient to symmetrize the problem by introducing

$$\psi_\alpha(x,\mu) = F_\alpha(x,\mu) + F_\alpha(-x,-\mu) \quad , \quad \alpha = 1 \text{ and } 2 \quad . \tag{4}$$

Thus, we seek solutions of

$$\mu \frac{\partial}{\partial x} \psi_\alpha(x,\mu) + \psi_\alpha(x,\mu) = \frac{1}{2} c_\alpha \int_{-1}^{1} \psi_\alpha(x,\mu')d\mu' \quad , \tag{5}$$

such that

$$\psi_\alpha(x,\mu) = \psi_\alpha(-x,-\mu) \quad , \tag{6a}$$

$$\psi_1(a,\mu) = \psi_2(a,\mu) \quad , \quad \mu \in (-1,1) \quad , \tag{6b}$$

and

$$\int_{-1}^{1} \psi_2(x,\mu)d\mu \to \exp(x/\eta_0) \quad , \quad \text{as} \quad x \to \infty \quad . \tag{6c}$$

We note that

$$\Delta = (4\eta_0 \sinh a/\eta_0)^{-1} \int_{-a}^{a} \int_{-1}^{1} \psi_1(x,\mu')d\mu' dx \quad . \tag{7}$$

In the following section, we sketch briefly the solution by three different methods, and in Sec. III we list our numerical results.

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[1] C. E. SIEWERT, K. NESHAT, and J. S. PHELPS III, *Nucl. Sci. Eng.*, **64**, 884 (1977).
[2] V. C. BOFFI, V. G. MOLINARI, and G. SPIGA, *Nucl. Sci. Eng.*, **66**, 424 (1978).
[3] M. M. R. WILLIAMS, *Proc. Phys. Soc.*, **85**, 413 (1965).
[4] M. M. R. WILLIAMS, *Br. J. Appl. Phys.*, **16**, 1841 (1965).

II. ANALYSIS

II.A. *The Method of Elementary Solutions*

For region 1, we can write the angular flux as

$$\psi_1(x,\mu) = A(\nu_0)[\phi_1(\nu_0,\mu)\exp(-x/\nu_0) + \phi_1(-\nu_0,\mu)\exp(x/\nu_0)] + \int_0^1 A(\nu)[\phi_1(\nu,\mu)\exp(-x/\nu) + \phi_1(-\nu,\mu)\exp(x/\nu)]d\nu \quad , \tag{8}$$

where we have used a standard notation[5] for the elementary solutions. For region 2, we write

$$\psi_2(x,\mu) = B(\eta_0)\phi_2(\eta_0,\mu)\exp(-x/\eta_0) + \phi_2(-\eta_0,\mu)\exp(x/\eta_0) + \int_0^1 B(\eta)\phi_2(\eta,\mu)\exp(-x/\eta)d\eta \quad , \quad x > a \quad . \tag{9}$$

Instead of the boundary condition given by Eq. (6b), we prefer to use the Chandrasekhar[6] S function to deduce the boundary condition

$$\psi_1(a,-\mu) = \exp(a/\eta_0)\phi_2(\eta_0,\mu)\frac{H_2(\mu)}{H_2(\eta_0)} + \frac{1}{2\mu}\int_0^1 S_2(\mu',\mu)\psi_1(a,\mu')d\mu' \quad , \quad \mu \in (0,1) \quad , \tag{10}$$

where $H_\alpha(\mu)$ is the H function corresponding to region α and

$$S_2(\mu',\mu) = \frac{c_2\mu\mu'}{\mu + \mu'} H_2(\mu')H_2(\mu) \quad . \tag{11}$$

In a manner similar to that discussed in Ref. 1, we can now enter Eq. (8) into Eq. (10) to obtain a singular integral equation that can readily be regularized and solved numerically by iteration. Since only the inhomogeneous terms of the resulting equations are different from Eqs. (7) and (8) of Ref. 1, we do not list them here explicitly.

II.B. *The C_N Method*

The complementary theorem[7] allows us to replace a problem defined in terms of a finite body by a related infinite-medium problem that has superficial sources located at what would be the boundary of the considered finite body. Using this theorem, we note that it is possible[8–10] to express the angular flux at the boundary as the solution of a Fredholm equation. Approximating the unknown angular flux at the surface by a polynomial of order N in the variable μ (the direction cosine of the propagating neutrons) and projecting the mentioned Fredholm equation on to a polynomial basis constitutes the C_N method.

For the considered problem, we apply the method successively to regions 1 and 2 to deduce the following system of Fredholm equations:

$$\psi^-(\mu) = \int_{-1}^0 G_2(a \to a^+;\ \mu' \to \mu)\psi^-(\mu')\mu'd\mu' + \int_0^1 G_2(a \to a;\ \mu' \to \mu)\psi^+(\mu')\mu'd\mu' + S(\mu) \quad , \quad \mu \in (-1,0) \quad , \tag{12a}$$

and

$$-\psi^+(\mu) = \int_{-1}^0 [G_1(a \to a;\ \mu' \to \mu) + G_1(-a \to a;\ -\mu' \to \mu)] \times \psi^-(\mu')\mu'd\mu' + \int_0^1 [G_1(a \to a^-;\ \mu' \to \mu) + G_1(-a \to a;\ -\mu' \to \mu)] \times \psi^+(\mu')\mu'd\mu' \quad , \quad \mu \in (0,1) \quad . \tag{12b}$$

Here,

$$\psi^+(\mu) = \psi_1(a,\mu) = \psi_2(a,\mu) \quad , \quad \mu \in (0,1) \quad , \tag{13a}$$

$$\psi^-(\mu) = \psi_1(a,\mu) = \psi_2(a,\mu) \quad , \quad \mu \in (-1,0) \quad , \tag{13b}$$

and $G_\alpha(x' \to x;\ \mu' \to \mu)$ is the angular flux at (x,μ) due to a monodirection plane source, in an infinite medium characterized by c_α, at (x',μ'). In addition, the function $S(\mu)$ is the angular flux at $x = a$, in an infinite medium characterized by c_2, corresponding to a current at infinity such that

$$\int_{-1}^1 \psi_2(x,\mu)d\mu \to \exp(x/\eta_0) \text{ as } x \to \infty \quad , \tag{14}$$

or explicitly

$$S(\mu) = \frac{1}{2} c_2 \frac{\eta_0}{\eta_0 + \mu} \exp(a/\eta_0) \quad . \tag{15}$$

If we now approximate $\psi^\pm(\mu)$ by writing

$$\psi^\pm(\mu) = \sum_{n=0}^N \psi_n^\pm \mu^n \quad , \tag{16}$$

and substitute Eq. (16) into Eqs. (12), we can then "take moments" of the resulting equations to obtain a system of algebraic equations for $\psi_n^\pm$, $n = 0, 1, 2, \ldots, N$. Making a neutron balance for the slab, we are able to write

$$j(a) = -\frac{1}{2}(1 - c_1)\int_{-a}^a \int_{-1}^1 \psi_1(x,\mu)d\mu dx \quad , \tag{17}$$

where $j(a)$ is the net current at the interface:

$$j(a) = \int_0^1 \psi^+(\mu)\mu d\mu + \int_{-1}^0 \psi^-(\mu)\mu d\mu \quad . \tag{18}$$

Thus, we are able to compute the flux distortion factor in the manner

$$\Delta = -(2\eta_0 \sinh a/\eta_0)^{-1}\frac{j(a)}{1 - c_1} \quad . \tag{19}$$

II.C. *The Integral Transform Method*

For the integral transform method,[11–13] we consider the total flux

$$\phi(x) = \int_{-1}^1 F(x,\mu)d\mu \quad , \tag{20}$$

where $F(x,\mu)$ is the solution of Eqs. (1) and (2). We find that $\phi(x)$ can be expressed as the limit as $x_0 \to \infty$ of the solution $\phi(x_0;x)$ to the integral form of the transport equation

$$\phi(x_0;x) = \frac{1}{2}Q_0(x_0)E_1(|x - x_0|) + \frac{1}{2}\int_{-\infty}^\infty c(y)E_1(|x - y|)\phi(x_0;y)dy \quad . \tag{21}$$

[5] K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts (1967).

[6] S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, New York and London (1950).

[7] K. M. CASE, F. de HOFFMANN, and G. PLACZEK, *Introduction to the Theory of Neutron Diffusion*, Vol. I, p. 129, U.S. Government Printing Office (1953).

[8] P. BENOIST and A. KAVENOKY, *Nucl. Sci. Eng.*, **32**, 225 (1968).

[9] A. KAVENOKY, "La méthode C_N de résolution de l'équation du transport," Doctorate Thesis, Paris (1973).

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[13] H. HEMBD, *Nucl. Sci. Eng.*, **40**, 224 (1970).

Here, $E_1(x)$ is the usual exponential integral, $x_0 > a$,

$$c(y) = c_1 \ , \ y \in (-a,a) \ , \text{ and } c(y) = c_2 \ , \ y > |a| \ , \tag{22}$$

and

$$Q_0(x_0) = \frac{c_2\eta_0}{(\eta_0^2 - 1)}[1 - \eta_0^2(1 - c_2)]\exp(x_0/\eta_0) \ . \tag{23}$$

If we let

$$\tilde{\phi}(x_0;\omega) = \int_{-\infty}^{\infty} \exp(-i\omega x)\phi(x_0;x)\,dx \ , \ \omega \in (-\infty,\infty) \ , \tag{24}$$

then we can take the Fourier transform of Eq. (21) to obtain

$$\tilde{\phi}(x_0;\omega) = Q_0(x_0)\exp(-ix_0\omega)\tilde{g}(\omega) + \frac{(c_1 - c_2)}{\pi}\tilde{g}(\omega)\int_{-\infty}^{\infty}\frac{\sin a(\omega - \eta)}{\omega - \eta}\tilde{\phi}(x_0;\eta)\,d\eta \ , \tag{25}$$

where

$$\tilde{g}(\omega) = \frac{\tan^{-1}\omega}{\omega - c_2\tan^{-1}\omega} \ . \tag{26}$$

We can use the Gegenbauer-Clebsch formula,[14]

$$\frac{1}{\pi}\frac{\sin a(\omega - \eta)}{\omega - \eta} = \sum_{n=0}^{\infty} Z_n^{(a)}(\omega)Z_n^{(a)}(\eta) \ , \tag{27a}$$

or, in terms of Bessel functions,

$$\frac{1}{\pi}\frac{\sin a(\omega - \eta)}{\omega - \eta} = \sum_{n=0}^{\infty}\left(n + \frac{1}{2}\right)\frac{J_{n+1/2}(a\omega)}{\sqrt{\omega}}\frac{J_{n+1/2}(a\eta)}{\sqrt{\eta}} \ , \tag{27b}$$

to write Eq. (25) as

$$\tilde{\phi}(x_0;\omega) = Q_0(x_0)\exp(-ix_0\omega)\tilde{g}(\omega) + (c_1 - c_2)\tilde{g}(\omega)\sum_{n=0}^{\infty}\xi_n(x_0)Z_n^{(a)}(\omega) \ , \tag{28}$$

where

$$\xi_n(x_0) = \int_{-\infty}^{\infty} Z_n^{(a)}(\omega)\tilde{\phi}(x_0;\omega)\,d\omega \ . \tag{29}$$

We note that the terms of Eq. (28) can be inverted to yield the desired solution, namely,

$$\phi(x_0;x) = Q_0(x_0)g(x - x_0) + (c_1 - c_2)\sum_{n=0}^{\infty}(-1)^{n/2}\left(\frac{2n + 1}{4\pi a}\right)^{1/2} \times \xi_n(x_0)\int_{-a}^{a} g(x - y)P_n\left(\frac{y}{a}\right)dy \ , \tag{30}$$

where $P_n(x)$ denotes the n'th Legendre polynomial and

$$g(x) = \frac{1}{c_2\eta_0}\frac{\eta_0^2 - 1}{[1 - \eta_0^2(1 - c_2)]}\exp(-|x|/\eta_0) + \frac{1}{2}\int_0^1 \frac{\exp(-|x|/\nu)}{(1 - c_2\nu\tanh^{-1}\nu)^2 + \left(\frac{c_2\pi\nu}{2}\right)^2}\frac{d\nu}{\nu} \ . \tag{31}$$

Now, letting x_0 tend to infinity in Eq. (30), we find

$$\phi(x) = \exp(x/\eta_0) + (c_1 - c_2)\sum_{n=0}^{\infty}(-1)^{n/2}\left(\frac{2n + 1}{4\pi a}\right)^{1/2} \times \xi_n\int_{-a}^{a} g(x - y)P_n\left(\frac{y}{a}\right)dy \ , \tag{32}$$

where we use ξ_n to denote $\xi_n(\infty)$.

[14] N. WATSON, *Bessel Functions*, Cambridge University Press (1945).

TABLE I

Numerical Results for Selected Cases

c_1	c_2	a	Δ	Δ^*
0.1	0.8	1.0	0.3587	0.1713
0.3	0.8	1.0	0.4381	0.2090
0.5	0.8	1.0	0.5638	0.2688
0.7	0.8	1.0	0.7940	0.3780
0.9	0.8	1.0	1.3539	0.6436
0.1	0.95	0.005	0.9493	0.9463
0.3	0.95	0.005	0.9608	0.9578
0.5	0.95	0.005	0.9725	0.9695
0.8	0.95	0.005	0.9907	0.9876
0.5	0.95	0.05	0.8131	0.7923
0.5	0.95	0.1	0.7010	0.6678
0.5	0.95	0.25	0.5136	0.4595
0.5	0.95	0.5	0.3726	0.3027
0.5	0.95	1.0	0.2601	0.1771
0.5	0.95	2.0	0.1882	0.0938
0.1	0.95	0.3	0.3240	0.2842
0.1	0.975	0.3	0.2492	0.2277

If now we project Eq. (28) over the denumerable sequence $\{Z_n^{(a)}(\omega)\}$ and let x_0 tend to infinity, we find the system of linear algebraic equations,

$$\xi_m = (c_1 - c_2)\sum_{n=0}^{\infty}\mathbf{A}_{m,n}\xi_n + B_m \ , \ m = 0, 1, 2, \ldots \ . \tag{33}$$

The matrix elements $\mathbf{A}_{m,n}$ are the same as those used previously[2] for the constant-source case, and

$$B_m = (-1)^{-m/2}\left[\frac{\pi(2m + 1)}{a}\right]^{1/2}\int_{-a}^{a} P_m\left(\frac{x}{a}\right)\exp(x/\eta_0)\,dx \ . \tag{34}$$

With all of the results established, we can express the flux distortion factor defined by Eq. (3) as

$$\Delta = \left(\frac{a}{\pi}\right)^{1/2}\frac{\xi_0}{2\eta_0\sinh(a/\eta_0)} \ . \tag{35}$$

It is clear that Eq. (33) can be approximated by a finite number of equations and thus solved to yield ξ_0 and the desired Δ.

III. RESULTS

Numerical calculations were carried out using all three of the methods discussed in the previous section, and the results are given in Table I. Because each of the three methods used yielded the values given in the table, we believe our results to be accurate to the degree indicated. We include in Table I our previously reported[1,2] values of Δ^*, the flux-depression factor for the constant-source problem.

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