THE F_N METHOD FOR SOLVING RADIATIVE-TRANSFER PROBLEMS IN PLANE GEOMETRY

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Abstract. The F_N method of solving problems in radiative transfer for plane-parallel media with anisotropic scattering is established. The method utilizes properties of the 'exact' solution and leads to final equations that are particularly concise and easy to use.

1. Introduction

In two recent papers (Siewert and Benoist, 1978; Grandjean and Siewert, 1978) the F_N method ($F \Rightarrow$ *facile*) for isotropic scattering was established and used to calculate several basic quantities of interest in the theory of neutron diffusion. Because we found the method to be particularly tractable, easy to use for numerical calculations and very efficient in regard to computer-time requirements, we wish here to extend the method to include a more realistic phase function. We consider the equation of transfer, written as

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} (2l + 1) f_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu') \, d\mu', \tag{1}$$

where ω is the single-scattering albedo and the f_l (with $f_0 = 1$) are the coefficients in a Legendre expansion of the phase function. In addition, τ is the optical variable, μ is the direction cosine (with respect to the *positive* τ-axis) of the propagating radiation and, of course, $I(\tau, \mu)$ is the radiation intensity. For an atmosphere of optical thickness τ_0, we seek a solution to Equation (1) subject to the boundary conditions

$$I(0, \mu) = f_1(\mu), \qquad \mu > 0, \tag{2a}$$

and

$$I(\tau_0, -\mu) = f_2(\mu), \qquad \mu > 0, \tag{2b}$$

where $f_1(\mu)$ and $f_2(\mu)$ are considered specified. Note that here $I(\tau, \mu)$ is the total intensity and not just the diffuse field. It is clear that here we are dealing with boundary conditions that represent the part of a more general problem that does not depend on the azimuthal angle.

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2. Analysis

Since the work of Case (1960) and Mika (1961) there has been no real problem in establishing a rather general exact solution of Equation (1); for example, if we use the notation of McCormick and Kuščer (1973) we can write

$$I(\tau, \mu) = \sum_{\beta=0}^{\kappa-1} \left[A(\nu_\beta)\phi(\nu_\beta, \mu)\, e^{-\tau/\nu_\beta} + A(-\nu_\beta)\phi(-\nu_\beta, \mu)\, e^{\tau/\nu_\beta} \right] + \\ + \int_{-1}^{1} A(\nu)\phi(\nu, \mu)\, e^{-\tau/\nu}\, d\nu, \tag{3}$$

where we have κ $\pm$ pairs of zeros ($\pm\ \nu_\beta$) of

$$\Lambda(z) = 1 + \frac{\omega}{2} z \int_{-1}^{1} g(\mu, \mu) \frac{d\mu}{\mu - z}. \tag{4}$$

We note (cf. McCormick and Kuščer, 1973) that

$$\phi(\nu_\beta, \mu) = \frac{\omega}{2} \nu_\beta g(\nu_\beta, \mu) \left(\frac{1}{\nu_\beta - \mu} \right), \tag{5a}$$

$$\phi(\nu, \mu) = \frac{\omega}{2} \nu g(\nu, \mu) Pv \left(\frac{1}{\nu - \mu} \right) + \lambda(\nu)\, \delta(\nu - \mu), \tag{5b}$$

and

$$\lambda(\nu) = 1 + \frac{\omega}{2} \nu P \int_{-1}^{1} g(\mu, \mu) \frac{d\mu}{\mu - \nu}, \tag{6}$$

where

$$g(\nu, \mu) = \sum_{l=0}^{L} (2l + 1) f_l g_l(\nu) P_l(\mu). \tag{7}$$

In addition, $P_l(\mu)$ is used to denote Legendre's polynomial and the polynomials $g_l(\mu)$, of order l, are those introduced by Chandrasekhar (1950) – e.g.,

$$g_0(\nu) = 1, \tag{8a}$$

$$g_1(\nu) = h_0 \nu, \tag{8b}$$

$$g_2(\nu) = \tfrac{1}{2}(h_0 h_1 \nu^2 - 1), \tag{8c}$$

and, in general,

$$(l + 1) g_{l+1}(\nu) = \nu h_l g_l(\nu) - l g_{l-1}(\nu), \tag{9}$$

with

$$h_l = (2l + 1)(1 - \omega f_l). \tag{10}$$

When it comes to solving a boundary problem with the solution given by Equation (3) there generally is no difficulty when the full-range expansion theory is required (McCormick and Kuščer, 1973). However, for the considerably more interesting cases of half spaces on finite slabs the half-range theory (op. cit.) is usually required to proceed with an 'exact' solution. Here we use the full-range orthogonality theorem;

that is,

$$\int_{-1}^{1} \mu\phi(\xi, \mu)\phi(\xi', \mu)\, d\mu = 0, \qquad \xi \neq \xi', \tag{11}$$

along with Equation (3) to develop 'exact' singular integral equations for the required exit distributions, $I(0, -\mu)$, $\mu > 0$, and $I(\tau_0, \mu)$, $\mu > 0$. Thus, if we multiply Equation (3) evaluated at $\tau = 0$ and $\tau = \tau_0$ by $\mu\phi(-\xi, \mu)$, $\xi \in P = \{\nu_\beta\} \cup (0, 1)$, and integrate over μ from -1 to 1, we find that

$$\int_{-1}^{1} \mu\phi(-\xi, \mu)I(0, \mu)\, d\mu = A(-\xi)N(-\xi), \qquad \xi \in P, \tag{12a}$$

and

$$\int_{-1}^{1} \mu\phi(-\xi, \mu)I(\tau_0, \mu)\, d\mu = A(-\xi)N(-\xi)\, e^{\tau_0/\xi}, \qquad \xi \in P. \tag{12b}$$

In a similar manner we can multiply Equation (3) by $\mu\phi(\xi, \mu)$ and integrate to obtain

$$\int_{-1}^{1} \mu\phi(\xi, \mu)I(0, \mu)\, d\mu = A(\xi)N(\xi), \qquad \xi \in P, \tag{13a}$$

and

$$\int_{-1}^{1} \mu\phi(\xi, \mu)I(\tau_0, \mu)\, d\mu = A(\xi)N(\xi)\, e^{-\tau_0/\xi}, \qquad \xi \in P, \tag{13b}$$

where $N(\pm\xi)$ are normalization constants that can be eliminated between Equations (12) and (13) to yield our basic system of singular integral equations and constraints – that is,

$$\int_{-1}^{1} \mu\phi(-\xi, \mu)I(0, \mu)\, d\mu = e^{-\tau_0/\xi}\int_{-1}^{1} \mu\phi(-\xi, \mu)I(\tau_0, \mu)\, d\mu, \qquad \xi \in P, \tag{14a}$$

and

$$\int_{-1}^{1} \mu\phi(\xi, \mu)I(\tau_0, \mu)\, d\mu = e^{-\tau_0/\xi}\int_{-1}^{1} \mu\phi(\xi, \mu)I(0, \mu)\, d\mu, \qquad \xi \in P. \tag{14b}$$

We can use Equations (2) and the fact that $\phi(-\xi, \mu) = \phi(\xi, -\mu)$ to write Equations (14) in the more convenient form

$$\int_{0}^{1} \mu\phi(\xi, \mu)I(0, -\mu)\, d\mu + e^{-\tau_0/\xi}\int_{0}^{1} \mu\phi(-\xi, \mu)I(\tau_0, \mu)\, d\mu = L_1(\xi) \tag{15a}$$

and

$$\int_{0}^{1} \mu\phi(\xi, \mu)I(\tau_0, \mu)\, d\mu + e^{-\tau_0/\xi}\int_{0}^{1} \mu\phi(-\xi, \mu)I(0, -\mu)\, d\mu = L_2(\xi), \tag{15b}$$

where the two known functions are

$$L_1(\xi) = \int_0^1 \mu\phi(-\xi, \mu) f_1(\mu)\, d\mu + e^{-\tau_0/\xi} \int_0^1 \mu\phi(\xi, \mu) f_2(\mu)\, d\mu \tag{16a}$$

and

$$L_2(\xi) = \int_0^1 \mu\phi(-\xi, \mu) f_2(\mu)\, d\mu + e^{-\tau_0/\xi} \int_0^1 \mu\phi(\xi, \mu) f_1(\mu)\, d\mu. \tag{16b}$$

We note that if there were an inhomogeneous source term in Equation (1), then a particular solution could be added to the solution given by Equation (3) and thus only $L_1(\xi)$ and $L_2(\xi)$ in Equations (16) would be changed (see Siewert and Benoist, 1978).

The basic idea of the F_N method is that we do not attempt to solve the system of singular integral equations given by Equations (15), but rather we approximate $I(0, -\mu)$ and $I(\tau_0, \mu)$ in Equations (15) by the polynomials

$$I(0, -\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \qquad \mu > 0, \tag{17a}$$

and

$$I(\tau_0, \mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \qquad \mu > 0, \tag{17b}$$

to obtain

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi) + e^{-\tau_0/\xi} \sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi) = \frac{2}{\omega\xi} L_1(\xi), \qquad \xi \in P, \tag{18a}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha B_\alpha(\xi) + e^{-\tau_0/\xi} \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi) = \frac{2}{\omega\xi} L_2(\xi), \qquad \xi \in P, \tag{18b}$$

where

$$B_\alpha(\xi) = \frac{2}{\omega\xi} \int_0^1 \mu^{\alpha+1} \phi(\xi, \mu)\, d\mu \tag{19a}$$

and

$$A_\alpha(\xi) = \frac{2}{\omega\xi} \int_0^1 \mu^{\alpha+1} \phi(-\xi, \mu)\, d\mu. \tag{19b}$$

Using Equations (5), we find that the required functions $B_\alpha(\xi)$ and $A_\alpha(\xi)$ can readily be generated (for $\alpha \geqslant 0$) from the recursive relations

$$B_{\alpha+1}(\xi) = \xi B_\alpha(\xi) - \sum_{l=0}^{L} (2l + 1) f_l g_l(\xi)\, \Delta_{\alpha,l}, \tag{20a}$$

and

$$A_{\alpha+1}(\xi) = -\xi A_\alpha(\xi) + \sum_{l=0}^{L} (2l + 1) f_l (-1)^l g_l(\xi)\, \Delta_{\alpha,l}, \tag{20b}$$

where

$$\Delta_{\alpha,l} = \int_0^1 \mu^{\alpha+1} P_l(\mu)\, d\mu, \tag{21}$$

clearly can be integrated analytically. To begin the calculation of $B_\alpha(\xi)$ and $A_\alpha(\xi)$ we find we can write

$$B_0(\xi) = \frac{2}{\omega} - 1 - \frac{2}{\omega}\xi\psi(\xi)\log\left(1 + \frac{1}{\xi}\right) + \sum_{l=1}^{L}(2l+1)f_l g_l(\xi)\Pi_l(\xi) \tag{22a}$$

and

$$A_0(\xi) = 1 - \frac{2}{\omega}\xi\psi(\xi)\log\left(1 + \frac{1}{\xi}\right) + \sum_{l=1}^{L}(2l+1)f_l g_l(\xi)\Pi_l(\xi), \tag{22b}$$

where the characteristic function is given by

$$\psi(\mu) = \frac{\omega}{2}\sum_{l=0}^{L}(2l+1)f_l g_l(\mu)P_l(\mu); \tag{23}$$

and the Π-polynomials can be generated (for $l \geqslant 0$) from

$$(2l+1)\xi\Pi_l(\xi) = (-1)^l(2l+1)\,\Delta_{0,l} + (l+1)\Pi_{l+1}(\xi) + l\Pi_{l-1}(\xi), \tag{24}$$

with

$$\Pi_0(\xi) = 1, \tag{25a}$$

$$\Pi_1(\xi) = \xi - \tfrac{1}{2} \tag{25b}$$

and

$$\Pi_2(\xi) = \tfrac{3}{2}\xi(\xi - \tfrac{1}{2}). \tag{25c}$$

It is apparent that the basic functions $B_\alpha(\xi)$ and $A_\alpha(\xi)$ are comprised only of polynomials and the function $\log(1 + 1/\xi)$, and thus they can readily be computed numerically.

Since $B_\alpha(\xi)$ and $A_\alpha(\xi)$ are established we now wish to find from Equations (18) the constants a_α and b_α required in Equations (17). Clearly we can select $N + 1$ different values of $\xi \in P$, say $\{\xi_j\}$, and solve the following system of $2(N + 1)$ linear algebraic equations for a_α and b_α, $\alpha = 0, 1, 2, \ldots, N$:

$$\sum_{\alpha=0}^{N}\left[a_\alpha B_\alpha(\xi_j) + e^{-\tau_0/\xi_j}\, b_\alpha A_\alpha(\xi_j)\right] = \frac{2}{\omega\xi_j}L_1(\xi_j) \tag{26a}$$

and

$$\sum_{\alpha=0}^{N}\left[b_\alpha B_\alpha(\xi_j) + e^{-\tau_0/\xi_j}\, a_\alpha A_\alpha(\xi_j)\right] = \frac{2}{\omega\xi_j}L_2(\xi_j). \tag{26b}$$

3. Discussion

Clearly the manner in which the $N + 1$ points $\{\xi_j\}$ are chosen from $\xi \in P$ can affect the accuracy of the method. In our numerical studies to date (Grandjean and Siewert, 1978; Devaux and Grandjean, 1978) we have elected to use a simple scheme that keeps the points ξ_j far from each other; for example, when $\psi(\mu) > 0$ there is only one pair of discrete roots $\pm\nu_0$ and thus we have used (Grandjean and Siewert, 1978; Devaux and Grandjean, 1978) $\xi_0 = \nu_0$ for the F_0 approximation, $\xi_0 = \nu_0$ and $\xi_1 = 0$ for the F_1 approximation, $\xi_0 = \nu_0$, $\xi_1 = 0$ and $\xi_2 = 1$ for the F_2 approximation and, in general, $\xi_0 = \nu_0$ and the remaining ξ_j spaced equally distant in the interval [0, 1] for the F_N approximation. Extensive numerical studies have been reported for isotropic scattering (Grandjean and Siewert, 1978) and some results for $L = 1$ and also for a generalization of the Rayleigh scattering model ($L = 2$) have been established (Devaux and Grandjean, 1978). It seems that the F_5 approximation is capable of predicting values of the spherical albedo and the transmission factor accurate to four significant figures for $\tau_0 \geqslant 0.5$ and for many cases the F_3 approximation yields four-figure accuracy. A more detailed study of the accuracy of the F_N method for higher-order anisotropic scattering is soon to be undertaken (cf. Devaux and Grandjean, 1978).

We note that the F_N method yields immediately the generally most important quantities $I(0, -\mu)$, $\mu > 0$, and $I(\tau_0, \mu)$ $\mu > 0$. On the other hand, the complete solution at any τ is readily available from Equation (3) since we can use the full-range orthogonality theorem to find the expansion coefficients $A(\pm\xi)$ once we know, for example, $I(0, \mu)$, $\mu \in (-1, 1)$. Thus we can write

$$A(\xi) = \frac{1}{N(\xi)}\left[\int_0^1 \mu\phi(\xi, \mu)f_1(\mu)\,\mathrm{d}\mu - \frac{\omega}{2}\,\xi\sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi)\right] \tag{27a}$$

and

$$A(-\xi) = \frac{1}{N(-\xi)}\left[\int_0^1 \mu\phi(-\xi, \mu)f_1(\mu)\,\mathrm{d}\mu - \frac{\omega}{2}\,\xi\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi)\right], \tag{27b}$$

where (cf. Mika, 1961)

$$N(\pm\nu_\beta) = \pm\tfrac{1}{2}\omega\nu_\beta^2 g(\nu_\beta, \nu_\beta)\Lambda'(\nu_\beta) \tag{28}$$

and

$$N(\pm\nu) = \pm\nu[\lambda^2(\nu) + \tfrac{1}{4}\pi^2\omega^2\nu^2 g^2(\nu, \nu)]. \tag{29}$$

In conclusion we note that (*i*) the F_N method is considerably simpler and more efficient than the traditional spherical harmonics method, where, for example, for order N one usually begins a problem by looking for the zeros of the polynomial $g_{N+1}(\xi)$ – this already is more difficult then computing the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ required in the F_N method; (ii) the method clearly can be used for polarization studies, or for the multiband model, where coupled systems of equations similar to Equation (1) must be considered; (iii) for slabs of very small optical thickness – i.e. $\tau_0 < 0.5$ – we believe the method can be improved by considering the diffuse field separately; and (iv) as demonstrated for isotropic scattering (cf. Grandjean and Siewert, 1978), the F_N method also yields good results for two-media problems, and we expect it to be useful for multilayer problems.

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