A new approach to the inverse problem

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The full-range orthogonality theorem concerning the elementary solutions of the equation of transfer is used to develop a solution of the inverse problem for a finite plane-parallel slab.

I. INTRODUCTION

The solution of the inverse problem given recently[1] for a finite slab is an improvement over previous infinite-medium results[2–4]; however, because the reported solution[1] depends on spatial moments of the total flux, more improvement is sought. From an experimental point of view a solution in terms only of surface quantities is what is most desired. Here we wish to report a method by which the desired solution can be established.

II. ISOTROPIC SCATTERING

To study first the simplest inverse problem for radiative transfer or neutron diffusion in a finite slab we consider the equation of transfer

$$\mu\frac{\partial}{\partial x}\psi(x,\mu)+\psi(x,\mu)=\frac{\omega}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu',\quad x\in[-a,a],\tag{1}$$

where $\psi(x,\mu)$ is the angular flux, x is the position variable measured in mean-free paths, and ω is the single scattering albedo. Here for a transport problem defined by the boundary conditions

$$\psi(-a,\mu)=f_1(\mu),\quad \mu>0,\tag{2a}$$

and

$$\psi(a,-\mu)=f_2(\mu),\quad \mu>0,\tag{2b}$$

we wish to find ω in terms of $f_1(\mu), f_2(\mu)$ and the exit distributions $\psi(-a,-\mu)$ and $\psi(a,\mu)$, $\mu>0$. This problem was solved recently[5] for $f_1(\mu)=\mu^\beta, \beta=0,1,2,\cdots$, and $f_2(\mu)=0$; however, here we intend to develop a solution to the general problem and to establish a procedure that can readily be generalized to the case of anisotropic scattering.

We begin by expressing $\psi(x,\mu)$ in terms of Case's elementary solutions,[6] i.e.,

$$\psi(x,\mu)=A(\nu_0)\phi(\nu_0,\mu)e^{-x/\nu_0}+A(-\nu_0)\phi(-\nu_0,\mu)e^{x/\nu_0}+\int_{-1}^{1}A(\nu)\phi(\nu,\mu)e^{-x/\nu}\,d\nu,\tag{3}$$

where

$$\phi(\nu_0,\mu)=\frac{\omega}{2}\nu_0\frac{1}{\nu_0-\mu},\tag{4a}$$

$$\phi(\nu,\mu)=\frac{\omega\nu}{2}Pv\left(\frac{1}{\nu-\mu}\right)+\lambda(\nu)\delta(\nu-\mu),\tag{4b}$$

$$\lambda(\nu)=1-\omega\nu\tanh^{-1}(\nu),\tag{5}$$

and $\pm\nu_0$ are the zeros of

$$\Lambda(z)=1+\frac{\omega}{2}z\int_{-1}^{1}\frac{d\mu}{\mu-z}.\tag{6}$$

In Eq. (3), $A(\nu_0), A(-\nu_0)$, and $A(\nu)$ are expansion coefficients to be determined by the boundary conditions. We shall not need them here. We can now use the full-range orthogonality theorem[6]

$$\int_{-1}^{1}\phi(\xi,\mu)\phi(\xi',\mu)\mu\,d\mu=0,\quad \xi\neq\xi',\tag{7}$$

to deduce from Eq. (3) that

$$\int_{-1}^{1}\phi(-\xi,\mu)\psi(\pm a,\mu)\mu\,d\mu=A(-\xi)N(-\xi)e^{\pm a/\xi},\quad \xi\in P,\tag{8}$$

and

$$\int_{-1}^{1}\phi(\xi,\mu)\psi(\pm a,\mu)\mu\,d\mu=A(\xi)N(\xi)e^{\mp a/\xi},\quad \xi\in P,\tag{9}$$

where $N(\pm\xi)$ are normalization constants and $\xi\in P => \xi=\nu_0$ or $\xi=\nu\in(0,1)$. We can now eliminate $A(\pm\xi)\,N(\pm\xi)$ in Eqs. (8) and (9) to find the equations used recently by Siewert and Benoist[7] to develop the F_N method of solving problems in neutron diffusion, i.e.,

$$\int_{-1}^{1}\phi(\xi,\mu)\psi(-a,-\mu)\mu\,d\mu-e^{-2a/\xi}\times\int_{-1}^{1}\phi(\xi,\mu)\psi(a,-\mu)\mu\,d\mu=0,\quad \xi\in P,\tag{10a}$$

and

$$\int_{-1}^{1}\phi(\xi,\mu)\psi(a,\mu)\mu\,d\mu-e^{-2a/\xi}\times\int_{-1}^{1}\phi(\xi,\mu)\psi(-a,\mu)\mu\,d\mu=0,\quad \xi\in P.\tag{10b}$$

Equations (10) define a system of singular integral equations and constraints that can be used to deduce the exit distributions when ω is given; however, we can use Eqs. (10) here to find ω when we assume we can determine experimentally the exit distributions. Thus if we use Eq. (4b) in Eqs. (10) for $\xi=\nu\in(0,1)$ we can solve immediately for ω; from Eq. (10a) and Eq. (10b), respectively, we obtain

$$\omega=\frac{2}{k_1(\nu)}[\psi(-a,-\nu)-\psi(a,-\nu)e^{-2a/\nu}],\quad \nu\in(0,1),\tag{11a}$$

and

$$\omega=\frac{2}{k_2(\nu)}[\psi(a,\nu)-\psi(-a,\nu)e^{-2a/\nu}],\quad \nu\in(0,1),\tag{11b}$$

where

$$k_1(\nu)=\int_{-1}^{1}T(\mu,\nu)[\psi(-a,-\mu)-\psi(a,-\mu)e^{-2a/\nu}]\mu\,d\mu\tag{12a}$$

and

$$k_2(\nu)=\int_{-1}^{1}T(\mu,\nu)[\psi(a,\mu)-\psi(-a,\mu)e^{-2a/\nu}]\mu\,d\mu\tag{12b}$$

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with

$$T(\mu,\nu)=P\nu\left(\frac{1}{\mu-\nu}\right)+2\tanh^{-1}(\nu)\delta(\mu-\nu). \tag{13}$$

Clearly any value of $\nu\in(0,1)$ can be used in Eqs. (11) to give an explicit result for ω in terms of the exit distributions. The limit as $\nu\to 0$ is particularly interesting; we find

$$\omega=2\psi(-a,0^-)\left[\int_0^1\psi(-a,-\mu)\,d\mu+\int_0^1 f_1(\mu)\,d\mu\right]^{-1} \tag{14a}$$

and

$$\omega=2\psi(a,0^+)\left[\int_0^1\psi(a,\mu)\,d\mu+\int_0^1 f_2(\mu)\,d\mu\right]^{-1}. \tag{14b}$$

It is curious that Eqs. (14a) and (14b) involve, respectively, only the angular distribution at $x=-a$ and $x=a$. It is also obvious that other expressions for ω can be obtained from Eqs. (10) for $\xi=\nu\in(0,1)$; e.g., we can multiply Eqs. (10) for $\xi=\nu\in(0,1)$ by "arbitrary" functions, say $G_1(\nu)$ and $G_2(\nu)$, and integrate over ν to obtain

$$\omega=\frac{2\int_0^1 G_1(\nu)[\psi(-a,-\nu)-\psi(a,-\nu)e^{-2a/\nu}]\,d\nu}{\int_0^1 G_1(\nu)k_1(\nu)\,d\nu} \tag{15a}$$

and

$$\omega=\frac{2\int_0^1 G_2(\nu)[\psi(a,\nu)-\psi(-a,\nu)e^{-2a/\nu}]\,d\nu}{\int_0^1 G_2(\nu)k_2(\nu)\,d\nu}. \tag{15b}$$

III. ANISOTROPIC SCATTERING

Let us now consider the extension of the method discussed in Sec. II to the general case of anisotropic scattering. We start with

$$\mu\frac{\partial}{\partial x}\psi(x,\mu)+\psi(x,\mu)=\frac{\omega}{2}\sum_{l=0}^{N}(2l+1)f_lP_l(\mu)\int_{-1}^1 P_l(\mu')\psi(x,\mu')\,d\mu', \tag{16}$$

$$\psi(-a,\mu)=f_1(\mu),\quad \mu>0, \tag{17a}$$

and

$$\psi(a,-\mu)=f_2(\mu),\quad \mu>0. \tag{17b}$$

Here $f_0=1$ and we seek to express $\omega, f_1, f_2, \ldots, f_N$ in terms of the surface quantities $\psi(\pm a,\mu)$. Following the notation of McCormick and Kuščer,[8] we can write

$$\psi(x,\mu)=\sum_{\alpha=0}^{\kappa-1}[A(\nu_\alpha)\phi(\nu_\alpha,\mu)e^{-x/\nu_\alpha}+A(-\nu_\alpha)\phi(-\nu_\alpha,\mu)e^{x/\nu_\alpha}]+\int_{-1}^1 A(\nu)\phi(\nu,\mu)e^{-x/\nu}\,d\nu, \tag{18}$$

where now we have $\kappa\pm$ pairs of zeros $(\pm\nu_\alpha)$ of

$$\Lambda(z)=1+\frac{\omega}{2}z\int_{-1}^1 g(\mu,\mu)\,\frac{d\mu}{\mu-z}. \tag{19}$$

We note[8] that

$$\phi(\nu_\alpha,\mu)=\frac{\omega}{2}\nu_\alpha g(\nu_\alpha,\mu)\left(\frac{1}{\nu_\alpha-\mu}\right), \tag{20a}$$

$$\phi(\nu,\mu)=\frac{\omega}{2}\nu g(\nu,\mu)P\nu\left(\frac{1}{\nu-\mu}\right)+\lambda(\nu)\delta(\nu-\mu), \tag{20b}$$

and

$$\lambda(\nu)=1+\frac{\omega}{2}\nu P\int_{-1}^1 g(\mu,\mu)\,\frac{d\mu}{\mu-\nu}, \tag{21}$$

where

$$g(\nu,\mu)=\sum_{l=0}^{N}(2l+1)f_lg_l(\nu)P_l(\mu). \tag{22}$$

In addition, $P_l(\mu)$ is used to denote Legendre's polynomial and the polynomials $g_l(\mu)$, of order l, are those introduced by Chandrasekhar.[9] Our task of determining ω and the coefficients f_l would be extremely simple were it not for the fact that the $g_l(\nu)$ depend on

$$h_l=(2l+1)(1-\omega f_l); \tag{23}$$

e.g.,

$$g_0(\nu)=1,\quad g_1(\nu)=h_0\nu,\quad g_2(\nu)=\tfrac{1}{2}(h_0h_1\nu^2-1), \tag{24}$$

and, in general,

$$(l+1)g_{l+1}(\nu)=\nu h_lg_l(\nu)-lg_{l-1}(\nu). \tag{25}$$

The full-range orthogonality relation concerning the elementary solutions is of the same form[8] as for the isotropic scattering case, i.e.,

$$\int_{-1}^1\phi(\xi,\mu)\phi(\xi',\mu)\mu\,d\mu=0,\quad \xi\neq\xi', \tag{26}$$

and thus we can readily generalize Eqs. (10) to obtain

$$\int_{-1}^1\phi(\xi,\mu)\psi(-a,-\mu)\mu\,d\mu-e^{-2a/\xi}\int_{-1}^1\phi(\xi,\mu)\psi(a,-\mu)\mu\,d\mu=0,\quad \xi\in P, \tag{27a}$$

and

$$\int_{-1}^1\phi(\xi,\mu)\psi(a,\mu)\mu\,d\mu-e^{2a/\xi}\int_{-1}^1\phi(\xi,\mu)\psi(-a,\mu)\mu\,d\mu=0,\quad \xi\in P, \tag{27b}$$

for the general case. Here $\xi\in P\Rightarrow\xi\in\{\nu_\alpha\}\cup(0,1)$. If we now enter Eq. (20b) into Eqs. (27) we obtain

$$\int_{-1}^1 A(\mu,\nu)[\psi(-a,-\mu)-\psi(a,-\mu)e^{-2a/\nu}]\mu\,d\mu=2[\psi(-a,-\nu)-\psi(a,-\nu)e^{-2a/\nu}] \tag{28a}$$

and

$$\int_{-1}^1 A(\mu,\nu)[\psi(a,\mu)-\psi(-a,\mu)e^{-2a/\nu}]\mu\,d\mu=2[\psi(a,\nu)-\psi(-a,\nu)e^{-2a/\nu}] \tag{28b}$$

where

$$A(\mu,\nu)=\sum_{l=0}^{N}(2l+1-h_l)g_l(\nu)\times\left[P_l(\mu)P\nu\left(\frac{1}{\mu-\nu}\right)+2\delta(\mu-\nu)Q_l(\nu)\right] \tag{29}$$

with

$$Q_l(\nu)=\tfrac{1}{2}P\int_{-1}^1 P_l(x)\frac{dx}{\nu-x}. \tag{30}$$

It is clear that $N+1$ values of $\nu\in(0,1)$ can be chosen to generate, from either Eq. (28a) or Eq. (28b), $N+1$ algebraic equations involving the $N+1$ unknowns; however, the equations are nonlinear! If we let

$$M_1(\mu,\nu)=\psi(-a,-\mu)-\psi(a,-\mu)e^{-2a/\nu} \tag{31a}$$

and

$$M_2(\mu,\nu)=\psi(a,\mu)-\psi(-a,\mu)e^{-2a/\nu}, \tag{31b}$$

then we can write Eqs. (28) as

$$\sum_{l=0}^{N}(2l+1-h_l)g_l(\nu)R_l^{(1)}(\nu)=2M_1(\nu,\nu), \quad \nu\in(0,1), \tag{32a}$$

and

$$\sum_{l=0}^{N}(2l+1-h_l)g_l(\nu)R_l^{(2)}(\nu)=2M_2(\nu,\nu), \quad \nu\in(0,1), \tag{32b}$$

where the known functions are

$$R_l^{(\alpha)}(\nu) = \int_0^1\left[P_l(\mu)Pv\left(\frac{1}{\mu-\nu}\right)+2Q_l(\nu)\delta(\mu-\nu)\right]M_\alpha(\mu,\nu)\mu\,d\mu + (-1)^l\int_0^1 P_l(\mu)M_\alpha(-\mu,\nu)\mu\,\frac{d\mu}{\mu+\nu}. \tag{33}$$

We can consider Eqs. (32) evaluated at selected values of $\nu\in(0,1)$, say $\{\nu_\beta\}$, or multiply the equations by a sequence of convenient functions, say $\{G_\beta(\nu)\}$, and integrate over ν to generate equations to be solved for the h_l. It is rather easy to see that it is sufficient to solve $2N$, for $N>0$, linear algebraic equations to establish the desired h_l. Although the inversion of a $2N\times 2N$ matrix is required here, it is clear that the inverse problem can be solved in this manner. One serious limitation to this solution is the fact that N must be specified before finding the various h_l. The solution given in Ref. 1 did not suffer from this fact, but it did require the flux at all x. Clearly what is desired here is an orthogonality relation that could be used with Eqs. (32) to extract the coefficients h_l. To date, such a relation has not been found.

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